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DEVELOPMENT OF AN ADVANCED CONTINUUM THEORY FOR COMPOSITE LAMINATES

Phase II Annual Report

Volume II (Attachments)

March 31, 1992

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A continuum theory with a micro-macro structure was developed for composite laminates that satisfies the traction and displacement continuity requirements at interfaces. The interlaminar stresses were included in the model in a natural way and without any ad hoc assumptions. The theory is best suited for thick multi-constituent laminates composed of several thin plies. The built-in micro-structure of the continuum model can account for the effects of curvature and geometric nonlinearity. A set of constitutive relations in terms of material properties of individual constituents was developed which is capable of modeling fiber orientation and stacking sequence. The theory was further expanded to include the effects of temperature where a set of coupled thermo-mechanical field equations with corresponding constitutive relations were derived. The field equations for linearized kinematics and flat geometries were obtained. Development of the theory for cylindrical and spherical geometries is underway.

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AN ALTERNATIVE DEVELOPMENT OF A CONTINUUM THEORY FOR COMPOSITE LAMINATES

Note: Section 1.0 of this part is the same as the Section 1.0 presented in Volume I

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2. Preliminaries. General background.

In this section we introduce the coordinate systems, and the notations which will be used in the subsequent development. We also record some relevant results from classical three-dimensional continuum mechanics.

2.1 Coordinate Systems

Let the points of a region $\mathcal{R}$ in a three dimensional Euclidean space be referred to a fixed right-handed rectangular Cartesian coordinate system x^i $(i = 1,2,3)$ and let η^i $(i = 1,2,3)$ be a general *convected* curvilinear coordinate system defined by the transformation

$$x^i = x^i(\eta^1,\eta^2,\eta^3) \tag{2.1}$$

We assume the above transformation is nonsingular in $\mathcal{R}$, i.e.,

$$\det(\frac{\partial x^i}{\partial \eta^i}) \neq 0 \tag{2.2}$$

This implies the existence of the unique inverse such that

$$\eta^i = \eta^i(x^1,x^2,x^3) \tag{2.3}$$

We recall that a convected coordinate system is normally defined in relation to a continuous body and moves continuously with the body throughout the motion of the body from one configuration to another.

Throughout this work, all Latin indices (subscripts or superscripts) take the values 1,2,3; all Greek indices (subscripts or superscripts) take the values 1,2 and the usual summation convention is employed. We will use a comma for partial differentiation with respect to either space or surface coordinates such as η^i or η^α and a superposed dot for material time derivative, i.e., differentiation with respect to time holding the material coordinates, such as η^i or η^α, fixed. Also,

we use a vertical bar (|) or a double vertical bar (||) for covariant differentiation in 2 and 3 dimensional spaces, respectively. Also, for convenience, often we set $\eta^3 = \xi$ and adopt the notation

$$\eta^i = (\eta^\alpha, \xi) \tag{2.4}$$

As it becomes clear shortly, in order to adequately represent the behavior of composite laminate we need to introduce a second system of convected coordinates. We will designate this latter system by θ^i (i = 1,2,3). In addition, in the course of derivation of various results for the composite laminate we will encounter covariant differentiation with respect to a coordinate system which corresponds to composite continuum. To denote this we will use a single boldfaced vertical bar (**|**). In what follows, when there is a possibility of confusion, quantities which represent the same physical/geometrical concepts will be denoted by the same symbol but with an added asterisk (*) for classical three dimensional continuum mechanics or an added hat (ˆ) for the micro-structure and no addition for composite laminate (macro-structure). For example, the mass densities of a body in the contexts of the classical continuum mechanics, the Cosserat surface (micro-structure) and the composite laminate (macro-structure) will be denoted by ρ^*, $\hat{\rho}$ and ρ, respectively.

3. Kinematics of micro- and macro-structures

We begin our discussion of the kinematical results by assuming that the position vector of a particle P^* of a representative element (micro-structure), i.e., $\mathbf{p}^*(\eta^\alpha,\xi,\theta^3,t)$ in the present configuration has the form

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,\theta^3,t) + \xi\mathbf{d}(\eta^\alpha,\theta^3,t) \tag{3.1}$$

The dual of (3.1) in a reference configuration is given by

$$\mathbf{P}^* = \mathbf{R}^*(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3) \tag{3.2}$$

If the reference configuration is taken to be the initial configuration at time t = 0, we obtain

$$\begin{aligned}\mathbf{p}^*(\eta^\alpha,\xi,\theta^3,0) &= \mathbf{r}(\eta^\alpha,\theta^3,0) + \xi\mathbf{d}(\eta^\alpha,\theta^3,0)\\ &= \mathbf{R}(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3)\\ &= \mathbf{P}(\eta^\alpha,\xi,\theta^3)\end{aligned} \tag{3.3}$$

The velocity vector $\mathbf{v}^*$ of the three-dimensional shell-like micro-structure at time t is given by

$$\mathbf{v}^* = \frac{\partial\mathbf{p}^*(\eta^\alpha,\xi,\theta^3,t)}{\partial t} = \dot{\mathbf{p}}^*(\eta^\alpha,\xi,\theta^3,t) \tag{3.4}$$

where a superposed dot denotes the material time derivative, holding η^i and θ^i fixed. From (3.1) and (3.4) we obtain

$$\mathbf{v}^* = \mathbf{v} + \xi\mathbf{w} \tag{3.5}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}\ ,\quad \mathbf{w} = \dot{\mathbf{d}} \tag{3.6}$$

From (3.1) we have

$$g^*_\alpha = a_\alpha + \xi \frac{\partial \mathbf{d}}{\partial \eta^\alpha} \quad , \quad g^*_3 = \mathbf{d} \tag{3.7}$$

where $\mathbf{a}_\alpha$ are the surface base vector of the surface s_o. The base vectors $g_i^*(\eta^\alpha,\xi,\theta^3,t)$ in (3.7) when evaluated on the surface $s_o : \xi = 0$ reduce to

$$\begin{aligned} g^*_\alpha(\eta^\gamma,0,\theta^3,t) &= \mathbf{a}_\alpha(\eta^\gamma,\theta^3,t) \\ g^*_3(\eta^\gamma,0,\theta^3,t) &= \mathbf{d}(\eta^\gamma,\theta^3,t) \end{aligned} \tag{3.8}$$

where $\mathbf{g}_i^*$ satisfy the condition

$$[g^*_1\ g^*_2\ g^*_3] \neq 0 \tag{3.9}$$

This restriction holds for all time and values of $\eta^i = \{\eta^\alpha,\xi\}$ and θ^3. In particular, it is valid for $\xi = 0$ so that by (3.9) we also have

$$[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{d}] \neq 0 \tag{3.10}$$

this condition implies that the director $\mathbf{d}$ cannot be tangent to the surface s_o.

We recall that the director $\mathbf{d}$ is a three-dimensional vector and it can be written as

$$\mathbf{d} = d_i \mathbf{g}^i = d^i \mathbf{g}_i \quad , \quad d_i = \mathbf{g}_i \cdot \mathbf{d} \quad , \quad d^i = g^{ij} d_j \tag{3.11}$$

where d_i and d^i denote the covariant and contravariant components of $\mathbf{d}$ referred to $\mathbf{g}^i$ and $\mathbf{g}_i$, respectively. The gradient of the director $\mathbf{d}$ may be obtained as follows:

$$\begin{aligned} \mathbf{d}_{,i} = (d^j \mathbf{g}_j)_{,i} = d^j{}_{,i}\mathbf{g}_j + d^j \mathbf{g}_{j,i} &= d^j{}_{,i}\mathbf{g}_j + d^j \{{}_i{}^k{}_j\} \mathbf{g}_k \\ &= d^j{}_{,i}\mathbf{g}_j + d^k \{{}_i{}^j{}_k\} \mathbf{g}_j \\ &= (d^j{}_{,i} + \{{}_i{}^j{}_k\} d^k) \mathbf{g}_j \\ &= d^j{}_{|i} \mathbf{g}_j \end{aligned} \tag{3.12}$$

where { } stands for the Christoffel symbol of the second kind and a vertical bar (|) denotes covariant differentiation with respect to g_{ij}. In obtaining (3.12) we have made use of the tensor identity

$$\mathbf{g}_{j,i} = \{_i{}^k{}_j\}\mathbf{g}_k \tag{3.13}$$

For convenience we introduce the notations

$$\lambda_{ij} = \mathbf{g}_i \cdot \mathbf{d}_{,j} = d_{i|j}$$
$$\lambda^i{}_j = \mathbf{g}^i \cdot \mathbf{d}_{,j} = d^i{}_{|j} \tag{3.14}$$

From (3.14) it is clear that

$$\lambda^i{}_j = g^{ik}\lambda_{kj} \tag{3.15}$$

Making use of (3.14) we may rewrite (3.12) as

$$\mathbf{d}_{,i} = \lambda_{ji}\mathbf{g}^j = \lambda^j{}_i\mathbf{g}_j \tag{3.16}$$

Consider now the velocity vector v which can be written in the form

$$\mathbf{v} = v^i\mathbf{g}_i = v_i\mathbf{g}^i \tag{3.17}$$

Since the coordinates θ^i are convected, it follows that

$$\mathbf{v}_{,i} = \mathbf{v}_{|j} = \dot{\mathbf{g}}_i \tag{3.18}$$

Following the same procedure used in (3.12), we can reduce (3.18) to

$$\mathbf{v}_{,i} = (v^j \mathbf{g}_j)_{,i} = v^j{}_{,i}\mathbf{g}_j + v^j \mathbf{g}_{j,i} = v^j{}_{,i}\mathbf{g}_j + \{{}_i{}^k{}_j\} v^j \mathbf{g}_k$$

$$= v^j{}_{,i}\mathbf{g}_j + \{{}_i{}^j{}_k\} v^k \mathbf{g}_j$$

$$= (v^j{}_{,i} + \{{}_i{}^j{}_k\} v^k)\mathbf{g}_j$$

$$= v^j{}_{|j}\mathbf{g}_j \tag{3.19}$$

where in obtaining (3.19) we have made use of (3.13) and (3.17). We now introduce the notations

$$v_{ij} = \mathbf{g}_i \cdot \mathbf{v}_{,j} = v_{i|j}$$
$$v^i{}_j = \mathbf{g}^i \cdot \mathbf{v}_{,j} = v^i{}_{|j} \tag{3.20}$$

From (3.20) it is clear that

$$v^i{}_j = g^{ik} v_{k|j} \tag{3.21}$$

Making use of (3.20), we may rewrite (3.19) as

$$\mathbf{v}_{,i} = v_{ji}\mathbf{g}^j = v^j{}_i\mathbf{g}_j \tag{3.22}$$

We observe that both λ_{ij} and v_{ij} represent the covariant derivative of vector components and hence transform as components of second order covariant tensors.

Since v_{ij} is a second order covariant tensor, we may decompose it into its symmetric and its skew-symmetric parts, i.e.,

$$v_{ij} = v_{(ij)} + v_{[ij]} = \eta_{ij} + \omega_{ij} \tag{3.23}$$

where

$$\eta_{ij} = v_{(ij)} = \frac{1}{2}(v_{ij} + v_{ji}) \tag{3.24}$$

and

$$\omega_{ij} = v_{[ij]} = \frac{1}{2}(v_{ij} - v_{ji}) \tag{3.25}$$

represent the symmetric and the skew-symmetric parts of v_{ij}, respectively. From (3.24) and (3.25), after making use of (3.18) and (3.20), we have

$$\eta_{ij} = \frac{1}{2}(v_{ij} + v_{ji}) = \frac{1}{2}(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j + \mathbf{g}_j \cdot \dot{\mathbf{g}}_i) = \frac{1}{2}\overline{(\mathbf{g}_i \cdot \mathbf{g}_j)}^{\cdot} = \frac{1}{2}\dot{g}_{ij} = \eta_{ji} \tag{3.26}$$

and

$$\omega_{ij} = \frac{1}{2}(v_{ij} - v_{ji}) = \frac{1}{2}(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j - \mathbf{g}_j \cdot \dot{\mathbf{g}}_i) = -\omega_{ji} \tag{3.27}$$

Also, in iew of (3.18) and (3.23), we may express $\dot{\mathbf{g}}_i$ in the form

$$\dot{\mathbf{g}}_i = \mathbf{v}_{,i} = (\eta_{ki} + \omega_{ki})\mathbf{g}^k \tag{3.28}$$

Moreover, the time rate of change of the determinant of g_{ij}, i.e., g is obtained as follows

$$\dot{g} = \overline{\det(g_{ij})}^{\cdot} = \frac{\partial}{\partial g_{kl}}(\det(g_{kj}))\dot{g}_{kl} = g g^{ij}\dot{g}_{ij} \tag{3.29}$$

where we have made use of the formula for the derivative of a determinant, namely

$$\frac{\partial}{\partial g_{kl}}(\det(g_{ij})) = g g^{kl} \tag{3.30}$$

Also, by making use of the relation

$$g^{ij}g_{kj} = \delta^i{}_j \tag{3.31}$$

we obtain an expression for $\dot{g}^{ij}$ as follows

$$\overline{(g^{ij}g_{kj})}^{\cdot} = \dot{\overline{\delta^i{}_j}} = 0$$

or

$$\dot{g}^{ij}g_{ki} = -\,g^{ik}\dot{g}_{kj}$$

or

$$\dot{g}^{ij}g_{kj}g^{jl} = -\,g^{jl}g^{ik}\dot{g}_{kj}$$

or

$$\dot{g}^{ik}\delta_k^l = -\,g^{ik}g^{lj}\dot{g}_{kj}$$

or

$$\dot{g}^{ij} = -\,g^{ik}g^{jl}\dot{g}_{kl} \tag{3.32}$$

Next, we proceed to obtain an expression for the director velocity **w**. Thus, we write

$$\begin{aligned}
\mathbf{w} = \dot{\mathbf{d}} &= w_k\mathbf{g}^k = w^k\mathbf{g}_k = \overline{(d_i\mathbf{g}^i)}^{\boldsymbol{\cdot}} \\
&= \dot{d}_i\mathbf{g}^i + d_i\dot{\mathbf{g}}^i = \dot{d}_i\mathbf{g}^i + d_i\overline{(g^{ij}\mathbf{g}_j)}^{\boldsymbol{\cdot}} \\
&= \dot{d}_k\mathbf{g}^k + d_i(\dot{g}^{ij}\mathbf{g}_j + g^{ij}\dot{\mathbf{g}}_j) \\
&= \dot{d}_k\mathbf{g}^k + d_i\{-\,g^{ik}g^{jl}\dot{g}_{kl}\mathbf{g}_j + g^{ij}(\eta_{kj} + \omega_{kj})\mathbf{g}^k\} \\
&= \dot{d}_k\mathbf{g}^k - d^kg^{jl}\dot{g}_{kl}\mathbf{g}_j + d^j\eta_{kj}\mathbf{g}^k + d^j\omega_{kj}\mathbf{g}^k \\
&= \dot{d}_k\mathbf{g}^k + d^i\omega_{ki}\mathbf{g}^k - d^i\dot{g}_{ik}\mathbf{g}^k + d^i\eta_{ki}\mathbf{g}^k \\
&= \dot{d}_k\mathbf{g}^k + d^i\omega_{ki}\mathbf{g}^k - d^i(2\eta_{ik})\mathbf{g}^k + d^i\eta_{ki}\mathbf{g}^k \\
&= \dot{d}_k\mathbf{g}^k + d^i(\omega_{ki} - \eta_{ki})\mathbf{g}^k
\end{aligned} \tag{3.33}$$

where in obtaining (3.33) we have made use of (3.28) and (3.32). The gradient of the director

velocity is obtained in a similar manner:

$$\mathbf{w}_{,i} = \dot{\mathbf{d}}_{,i} = \overline{(\mathbf{d}_k\mathbf{g}^k)_{,i}}^{\,\cdot} = \overline{(\lambda_{ki}\mathbf{g}^k)}^{\,\cdot} = \dot{\lambda}_{ki}\mathbf{g}^k + \lambda_{ki}\dot{\mathbf{g}}^k$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k + \lambda_{ki}\overline{(g^{kj}\mathbf{g}_j)}^{\,\cdot} = \dot{\lambda}_{ki}\mathbf{g}^k + \lambda_{ki}(\dot{g}^{kj}\mathbf{g}_j + g^{kj}\dot{\mathbf{g}}_j)$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k + \lambda_{ki}(-g^{km}g^{jl}\dot{g}_{ml}\mathbf{g}_j) + \lambda_{ki}g^{kj}(\eta_{mj} + \omega_{mj})\mathbf{g}^m$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k - \lambda^m{}_i\dot{g}_{ml}\mathbf{g}^l + \lambda^j{}_i(\eta_{mj} + \omega_{mj})\mathbf{g}^m$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k - \lambda^m{}_i(2\eta_{ml})\mathbf{g}^l + \lambda^j{}_i\eta_{kj}\mathbf{g}^k + \lambda^j{}_i\omega_{kj}\mathbf{g}^k$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k + \lambda^j{}_i\omega_{kj}\mathbf{g}^k - 2\lambda^j{}_i\eta_{jk}\mathbf{g}^k + \lambda^j{}_i\eta_{kj}\mathbf{g}^k$$

$$= \dot{\lambda}_{ki}\mathbf{g}^k + \lambda^j{}_i(\omega_{kj} - \eta_{kj})\mathbf{g}^k \tag{3.34}$$

The dual of expressions (3.7) to (3.16) in the reference configuration follows from (3.2) in a similar manner and is given by:

$$\mathbf{G}^*_\alpha = \mathbf{A}_\alpha + \xi\mathbf{D}_{,\alpha} \quad , \quad \mathbf{G}^* = \mathbf{D} \tag{3.35}$$

$$\begin{aligned} \mathbf{G}^*_\alpha(\eta^\gamma,0,\theta^3) &= \mathbf{A}_\alpha(\eta^\gamma,\theta^3) \\ \mathbf{G}^*_3(\eta^\gamma,0,\theta^3) &= \mathbf{D}(\eta^\gamma,\theta^3) \end{aligned} \tag{3.36}$$

where $\mathbf{G}^*_i$, $\mathbf{d}$ satisfy the conditions

$$[\mathbf{G}^*_1\ \mathbf{G}^*_2\ \mathbf{G}^*_3] \neq 0 \tag{3.37}$$

and

$$[\mathbf{G}^*_1\ \mathbf{G}^*_2\ \mathbf{D}] \neq 0 \tag{3.38}$$

Moreover,

$$D = D_i G^i = D^i G_i \quad , \quad D_i = G_i \cdot \mathbf{D} \quad , \quad D^i = G^{ij} D_i \tag{3.39}$$

$$\mathbf{D}_{,i} = D^j{}_{|i} G_j = \Lambda^j{}_i G_j = \Lambda_{ji} G^j \tag{3.40}$$

where we have

$$\begin{aligned} \Lambda_{ij} &= G_i \cdot \mathbf{D}_{,j} = D_{i\,|\,j} \\ \Lambda^i{}_j &= G^i \cdot \mathbf{D}_{,j} = D^i{}_{|\,j} \end{aligned} \tag{3.41}$$

and

$$\Lambda^i{}_j = G^{ik} \Lambda_{kj} \tag{3.42}$$

We now introduce relative kinematical measures γ_{ij}, $\mathcal{K}_{ij}$ and γ_i such that

$$\gamma_{ij} = \frac{1}{2}\,(g_{ij} - G_{ij}) = \frac{1}{2}\,(g_i \cdot g_j - G_i \cdot G_j) = \gamma_{ji} \tag{3.43}$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij} \tag{3.44}$$

and

$$\gamma_i = d_i - D_i \tag{3.45}$$

Making use of (3.7), (3.11), (3.16), (3.35), (3.39) and (3.40) we may obtain

$$\begin{aligned} \gamma^*_{\alpha\beta} = \gamma^*_{\beta\alpha} &= \frac{1}{2}\,\{(g_\alpha + \xi \mathbf{d}_{,\alpha}) \cdot (g_\beta + \xi \mathbf{D}_{,\beta}) - (G_\alpha + \xi \mathbf{D}_{,\alpha}) \cdot (G_\beta + \xi \mathbf{D}_{,\beta})\} \\ &= \frac{1}{2}\,\{(g_{\alpha\beta} - G_{\alpha\beta}) + \xi[(g_\alpha \cdot \mathbf{d}_{,\beta} - G_\alpha \cdot \mathbf{D}_{,\beta}) + (g_\beta \cdot \mathbf{d}_{,\alpha} - G_\beta \cdot \mathbf{D}_{,\alpha}) \\ &\quad + \xi^2(\mathbf{d}_{,\alpha} \cdot \mathbf{d}_{,\beta} - \mathbf{D}_{,\alpha} \cdot \mathbf{D}_{,\beta})\} \end{aligned}$$

or

BASE

$$\gamma^*_{\alpha\beta} = \gamma^*_{\beta\alpha} = \gamma_{\alpha\beta} = \frac{1}{2}\,\xi(\mathcal{K}_{\alpha\beta} + \mathcal{K}_{\beta\alpha}) + \frac{1}{2}\,\xi^2(\lambda^i{}_\alpha\lambda_{i\beta} - \Lambda^i{}_\alpha\Lambda_{i\beta}) \tag{3.46}$$

Also,

$$\gamma^*_{\alpha 3} = \gamma^*_{3\alpha} = \frac{1}{2}\{(\mathbf{g}_\alpha + x\mathbf{d}_{,\alpha}) \cdot \mathbf{d} - (\mathbf{G}_\alpha + \xi\mathbf{D}_{,\alpha}) \cdot \mathbf{D}\}$$

$$= \frac{1}{2}\,\{(\mathbf{g}_\alpha \cdot \mathbf{d} - \mathbf{G}_\alpha \cdot \mathbf{D}) + \xi(\mathbf{d} \cdot \mathbf{d}_{,\alpha} - \mathbf{D} \cdot \mathbf{D}_{,\alpha})\}$$

or

$$\gamma^*_{\alpha 3} = \gamma^*_{3\alpha} = \frac{1}{2}\,\{\gamma_\alpha + \xi(d^i\lambda_{i\alpha} - D^i\Lambda_{i\alpha})\} \tag{3.47}$$

and

$$\gamma^*_{33} = \frac{1}{2}\,(\mathbf{d} \cdot \mathbf{d} - \mathbf{D} \cdot \mathbf{D}) = \frac{1}{2}\,(d^i d_i - D^i D_i) \tag{3.48}$$

4. Basic field equations for the micro-structure

Making use of the theory of Cosserat (directed) surfaces after appropriate integration of the classical three-dimensional equations of motion, across the thickness of the micro-structure, we obtain the basic field equations for the shell-like micro-structure as follows:

$$
\begin{aligned}
&\mathrm{a}: \ \overline{(\hat{\rho} a^{1/2})}^{\boldsymbol{\cdot}} = 0 \\
&\mathrm{b}: \ \hat{\rho} a^{1/2}(\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) = (\mathbf{N}^\alpha a^{1/2})_{,\alpha} + \hat{\rho}\hat{\mathbf{f}} a^{1/2} \\
&\mathrm{c}: \ \hat{\rho} a^{1/2}(y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) = (\mathbf{M}^\alpha a^{1/2})_{,\alpha} - \mathbf{m} a^{1/2} + \hat{\rho}\hat{\mathbf{l}} a^{1/2} \qquad (4.1) \\
&\mathrm{d}: \ \mathbf{a}_\alpha \times \mathbf{N}^\alpha + \mathbf{d} \times \mathbf{m} + \mathbf{d}_{,\alpha} \times \mathbf{M}^\alpha = 0 \\
&\mathrm{e}: \ \hat{\rho}(\dot{\hat{\varepsilon}}) = \mathbf{N}^\alpha \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^\alpha \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w}
\end{aligned}
$$

where the various field quantities appear in (4.1) are

$\hat{\rho} = \hat{\rho}(\theta^\alpha, t)$:	The mass density of the micro-structure in the present configuration
$\nu = \nu(\theta^\alpha, t)$:	The outward unit normal to the boundary $\partial\hat{\mathcal{P}}$ of the microstructure
$\mathbf{N}^\alpha = \mathbf{N}^\alpha(\theta^\alpha, t; \nu)$:	The resultant force per unit length of a curve in the present configuration
$\mathbf{M}^\alpha = \mathbf{M}^\alpha(\theta^\alpha, t; \nu)$:	The resultant couple per unit length of a curve in the present configuration
$\hat{\mathbf{f}} = \hat{\mathbf{f}}(\theta^\alpha, t)$:	The assigned force per unit mass of the micro-structure
$\hat{\mathbf{l}} = \hat{\mathbf{l}}(\theta^\alpha, t)$:	The assigned director force per unit mass of the microstructure
$\mathbf{m} = \mathbf{m}(\theta^\alpha, t)$:	The intrinsic director force per unit area of the micro-structure

$y^\alpha = y^\alpha(\theta^\alpha)$: The inertia coefficients

$\hat{\varepsilon} = \hat{\varepsilon}(\theta^\alpha, t)$: The specific internal energy per unit mass of the micro-structure

The relations between the above field quantities and the field quantities in classical three-dimensional continuum mechanics are given below:

$$\hat{\rho} a^{1/2} = \int_0^{\xi_2} \rho^* g^{*1/2} d\xi = \int_0^{\xi_1} \rho_1^* g^{*1/2} d\xi + \int_{\xi_1}^{\xi_2} \rho_2^* g^{*1/2} d\xi \tag{4.2}$$

$$\hat{\rho} a^{1/2} y^\alpha = \int_0^{\xi_2} \rho^* g^{*1/2} \xi^\alpha d\xi = \int_0^{\xi_1} \rho_1^* g^{*1/2} \xi^\alpha d\xi + \int_{\xi_1}^{\xi_2} \rho_2^* g^{*1/2} \xi^\alpha d\xi \tag{4.3}$$

$$\mathbf{N}^\alpha a^{1/2} = \int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi = \int_0^{\xi_1} \mathbf{T}^{*\alpha} d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} d\xi \tag{4.4}$$

$$\mathbf{M}^\alpha a^{1/2} = \int_0^{\xi_2} \mathbf{T}^{*\alpha} \xi d\xi = \int_0^{\xi_1} \mathbf{T}^{*\alpha} \xi d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} \xi d\xi \tag{4.5}$$

$$\mathbf{m} a^{1/2} = \int_0^{\xi_2} \mathbf{T}^{*3} d\xi = \int_0^{\xi_1} \mathbf{T}^{*3} d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3} d\xi \tag{4.6}$$

$$\hat{\rho} \hat{\mathbf{f}} a^{1/2} = \int_0^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} d\xi + [\mathbf{T}^{*3}]_{\xi=0}^{\xi=\xi_2} \tag{4.7}$$

$$\hat{\rho} \hat{\mathbf{l}} a^{1/2} = \int_0^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} \xi d\xi + [\mathbf{T}^{*3} \xi]_{\xi=0}^{\xi=\xi_2} \tag{4.8}$$

The conservation laws for the micro-structure $\hat{\mathcal{P}}$, bounded by $\partial\hat{\mathcal{P}}$, may be obtained by integration of $(4.1)_a$ to $(4.1)_e$ over appropriate range of integration of the micro-structure (i.e., Cosserat surface). In this fashion we obtain

$$a : \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\, d\hat{a} = 0$$

$$b : \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{v} + y^1\mathbf{w})\, d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}\hat{\mathbf{f}}\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{N}\, ds$$

$$c : \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(y^1\mathbf{v} + y^2\mathbf{w})\, d\hat{a} = \int_{\hat{\mathcal{P}}} (\hat{\rho}\hat{\mathbf{l}} - \mathbf{m})\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{M}\, ds$$

$$d : \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}[\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})]\, d\hat{a} = \tag{4.9}$$

$$\int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}})\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{r} \times \mathbf{N} + \mathbf{d} \times \mathbf{M})\, ds$$

$$e : \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\varepsilon} + \hat{\mathcal{K}})\, d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{f} \cdot \mathbf{v} + \hat{\mathbf{l}} \cdot \mathbf{w})\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \mathbf{w})\, ds$$

where $\hat{\mathcal{K}}$ is the kinetic energy per unit mass of the micro-structure and $\hat{\mathcal{P}}$ is an arbitrary part of the Cosserat surface (i.e., micro-structure) with its boundary curve $\partial\hat{\mathcal{P}}$. The first of (4.9) is a mathematical statement of the conservation of mass, the second that of the linear momentum, the third is the conservation of the director momentum, the fourth that of the moment of momentum, and the fifth is the conservation of energy.

In (4.9) the micro-structure's contact force $\mathbf{N}$ and contact couple $\mathbf{M}$ (director force) are defined by

$$\int_{\hat{\mathcal{P}}} \mathbf{N}\, ds = \int_{\partial\mathcal{P}^{*\prime}} \mathbf{t}^* da \quad , \quad \int_{\hat{\mathcal{P}}} \mathbf{M}\, ds = \int_{\partial\mathcal{P}^{*\prime}} \mathbf{t}^*\xi\, d\xi \tag{4.10}$$

and are related to $\mathbf{N}^\alpha$ and $\mathbf{M}^\alpha$ as follows:

$$\mathbf{N} = \mathbf{N}^\alpha \nu_\alpha \quad , \quad \mathbf{M} = \mathbf{M}^\alpha \nu_\alpha \tag{4.11}$$

where ν_α $(\alpha = 1,2)$ are covariant components of ν.

5. Summary of basic principles for composite laminates

This section contains a summary of basic principles for composite laminates. Also included in this section is an explanation of various quantities appearing in the conservation laws. With reference to the present configuration, these conservation laws are:

$$
\begin{aligned}
&a: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho \, dv = 0 \\
&b: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{v} + y^1\mathbf{w})dv = \int_{\mathcal{P}} \rho \, \mathbf{b} \, dv + \int_{\partial\mathcal{P}} \mathbf{t} \, da \\
&c: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(y^1\mathbf{v} + y^2\mathbf{w})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s} \, da \qquad (5.1) \\
&d: \quad \frac{d}{dt}\int_{\mathcal{P}} \{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}dv = \\
&\qquad\qquad \int_{\mathcal{P}} \rho(\mathbf{r} \times \mathbf{b} + \mathbf{d} \times \mathbf{c})dv + \int_{\partial\mathcal{P}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s})da \\
&e: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(\varepsilon + \mathcal{K})dv = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w})dv + \int_{\partial\mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w})da
\end{aligned}
$$

The first of (5.1) is the mathematical statement of conservation of mass, the second that of linear momentum principle, the third that of director momentum, the fourth is the principle of moment of momentum, and the fifth represents the balance of energy for composite laminates.

In (5.1) $\mathbf{r}, \mathbf{d}$ denote the *position vector* and the *director* associated with a *composite particle*, respectively, while the velocity and the director velocity of the composite particle are given by $\mathbf{v}$ and $\mathbf{w}$. The definition of the various field quantities in (5.1) and their relation to their counterparts in micro-structure and the similar three dimensional quantities are given below.

1) $\rho = \rho(\theta^i, t)$ is the *composite assigned mass density* in the present configuration given by

$$\rho g^{1/2} = \hat{\rho} \, a^{1/2} = \frac{1}{\xi_2} \int_{o}^{\xi_2} \rho^* g^{*1/2} d\xi \qquad (5.2)$$

where in (5.2) $\hat{\rho}$ is the mass density of the micro-structure, ρ^* is the classical 3-dimensional mass density, g is the determinant of the metric tensor g_{ij} associated with the composite coordinate system θ^i, g^* is the determinant of the metric tensor g^*_{ij} associated with the micro-structure coordinate system $\eta^i = \{\eta^\alpha,\xi\} = \{\theta^\alpha,\xi\}$, a is the determinant of the two-dimensional (surface) metric tensor $a_{\alpha\beta}$ associated with the Cosserat surface (micro-structure).

We notice that the dimensions of ρ^* and $\hat{\rho}$ are mass per unit volume and mass per unit area, respectively. However, the dimension of ρ is the dimension of integrated mass per unit volume of the composite.

2) $\mathbf{b} = \mathbf{b}(\theta^i,t)$ is the *composite assigned body force density* per unit of ρ, given by

$$\rho\, g^{1/2}\, \mathbf{b} = \frac{1}{\xi_2} \int_o^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* d\xi \tag{5.3}$$

where $\mathbf{b}^*$ is the classical 3-dimensional body force density. The dimension of **b** should be clear from (5.3).

3) $\mathbf{c} = \mathbf{c}(\theta^i,t)$ is the *composite assigned body couple density* per unit of ρ, given by[4]

$$\rho\, g^{1/2}\, \mathbf{c} = \frac{1}{\xi_2} \int_o^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* \xi d\xi \tag{5.4}$$

The dimension of **c** should be clear from (5.4).

4) $\mathbf{t} = \mathbf{t}(\theta^i,t;\mathbf{n})$ is the *composite assigned stress vector* (per unit area of the composite) such that[5]

$$\mathbf{t} = g^{-1/2}\, \mathbf{T}^i n_i \tag{5.5}$$

---

[4] **c** may also be called "composite assigned director force" emphasizing the "directed" nature of the present continuum theory. In the present context, however, we prefer the terminology in 3 above as it makes the physical nature of **c** more apparent.

[5] The nature of the definition (5.5) and (5.6) as well as (5.9) and (5.10) will be discussed and explained in section (19).

$$T^i{}_{,i} = \frac{1}{\xi_2} \int_o^{\xi_2} T^{*i}{}_{,i} d\xi \tag{5.6}$$

$$T^\alpha = \frac{1}{\xi_2} \int_o^{\xi_2} T^{*\alpha} d\xi = \frac{1}{\xi_2} a^{1/2} N^\alpha \tag{5.7}$$

$$T^3{}_{,3} = \frac{1}{\xi_2} (T^{*3}{}_{|\xi=\xi_2} - T^{*3}{}_{|\xi=0}) = \frac{\Delta T^{*3}}{\xi_2} \tag{5.8}$$

where $\mathbf{T}^{*i}$ is the classical stress vector and $\mathbf{N}^\alpha$ is the resultant force of the micro-structure (i.e., Cosserat surface). We also recall that a comma on the left-hand side of (5.6) to (5.8) denotes partial differentiation with respect to θ^i. However, a comma on the right-hand side of (5.6) and in (5.8) denotes partial differentiation with respect to $\eta^i = \{\eta^\alpha, \xi\}$.

5) $\mathbf{s} = \mathbf{s}(\theta^i, t; \mathbf{n})$ is the *composite assigned couple stress vector* [6] per unit area of the composite such that

$$\mathbf{s} = g^{-1/2} \mathbf{S}^i n_i \tag{5.9}$$

$$S^i{}_{,i} = \frac{1}{\xi_2} \int_o^{\xi_2} T^{*i}{}_{,i} \xi d\xi \tag{5.10}$$

$$S^\alpha = \frac{1}{\xi_2} \int_o^{\xi} T^{*\alpha} \xi d\xi = \frac{1}{\xi_2} a^{1/2} M^\alpha \tag{5.11}$$

$$S^3{}_{,3} = \frac{1}{\xi_2} [(T^{*3}\xi)_{|\xi=\xi_2} - (T^{*3}\xi)_{|\xi=0}] = \frac{\Delta(T^{*3}\xi)}{\xi_2} \tag{5.12}$$

where $\mathbf{M}^\alpha$ is the resultant couple of the micro-structure (i.e., Cosserat surface) and the same remark as in (4) above holds for commas and partial differentiation.

6) $\mathbf{k} = \mathbf{k}(\theta^i, t)$ is the *composite assigned intrinsic (director) force*, per unit volume of the composite, given by

[6] s may also be called "composite assigned contact director force" which reflects the "directed" nature of the present theory. However, the terminology given in item 5 above reflects the physical nature of s more clearly.

$$g^{1/2}\mathbf{k} = \frac{1}{\xi_2}\, a^{1/2}\mathbf{m} = \frac{1}{\xi_2}\int_o^{\xi_2} \mathbf{T}^{*3} d\xi \tag{5.13}$$

where $\mathbf{m}$ is the intrinsic director force of the micro-structure (i.e., Cosserat surface).

7) $y^\alpha = y^\alpha(\theta^i)$ are the inertia coefficients which are independent of time and are given by

$$y^\alpha = \int_o^{\xi_2} \rho^* g^{*1/2} \xi^\alpha d\xi \tag{5.14}$$

8) $\varepsilon = \varepsilon(\theta^i, t)$ is the *composite assigned specific internal energy* per unit of ρ given by

$$\rho\, g^{1/2}\, \varepsilon = \frac{1}{\xi_2}\, \hat{\rho}\, a^{1/2}\, \hat{\varepsilon} = \frac{1}{\xi_2}\int_o^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi \tag{5.15}$$

where ε^* is the classical 3-dimensional specific internal energy and $\hat{\varepsilon}$ is the specific internal energy per unit $\hat{\rho}$ for the micro-structure (i.e., Cosserat surface).

9) $\mathcal{K} = \mathcal{K}(\theta^i, t)$ is the *composite assigned kinetic energy* per unit of ρ and is given by

$$\mathcal{K} = \hat{\mathcal{K}} = \frac{1}{2}\,(\mathbf{v}\cdot\mathbf{v} + 2y^1\mathbf{v}\cdot\mathbf{w} + y^2\mathbf{w}\cdot\mathbf{w}) \tag{5.16}$$

where $\hat{\mathcal{K}}$ represents the kinetic energy per unit $\hat{\rho}$ of the micro-structure (i.e., Cosserat surface). The momentum corresponding to the velocity $\mathbf{v}$ and the director momentum corresponding to $\mathbf{w}$ are given by

$$\rho\, \frac{\partial \mathcal{K}}{\partial \mathbf{v}} = \rho(\mathbf{v} + y^1\mathbf{w}) \tag{5.17}$$

$$\rho\, \frac{\partial \mathcal{K}}{\partial \mathbf{w}} = \rho(y^1\mathbf{v} + y^2\mathbf{w}) \tag{5.18}$$

For simplicity in the rest of this development, when there is no possibility of confusion, we adopt the following simplified terminology:

ρ: "composite mass density"

b: "composite body force density"

c: "composite body couple density"

t: "composite stress"

s: "composite couple stress"

k: "composite intrinsic force"

ε: "composite specific internal energy"

$\mathcal{K}$: "composite kinetic energy"

An important characteristic of the present theory is the introduction of the composite contact force, **t**, and the composite contact couple, s. It can be shown [4] that **t** and s have the properties

$$\mathbf{t}(\theta^i,t\,;\,\mathbf{n}) = -\,\mathbf{t}(\theta^i,t\,;\,-\mathbf{n}) \tag{5.19}$$

and

$$\mathbf{s}(\theta^i,t\,;\,\mathbf{n}) = -\,\mathbf{s}(\theta^i,t\,;\,\mathbf{n}) \tag{5.20}$$

where **n** is the outward unit normal to a surface within the composite. According to the results (5.19) and (5.20), the composite stress vector and the composite couple stress vector acting on opposite sides of the same surface at a given point within the composite laminates are equal in magnitude and opposite in direction. In addition, it can be demonstrated [4] that $\mathbf{T}^i$ and $\mathbf{S}^i$ are expressible as

$$\mathbf{T}^i = g^{1/2}\tau^{ij}\mathbf{g}_j \tag{5.21}$$

and

$$\mathbf{S}^i = g^{1/2}s^{ij}\mathbf{g}_j \tag{5.22}$$

where τ^{ij} and s^{ij} are contravariant components of the *composite stress tensor* and the *composite couple stress tensor.* It can also be shown that $\mathbf{T}^3$ represent the interlaminar stress vector which is related

to the three components of interlaminar stresses τ^{3j} (j = 1,2,3) through (5.22). While τ^{3j} represent the three components of stress tensor, the same is not true for the interlaminar couple stress tensors. In fact it can be shown that S^3 and s^{3j} (j = 1,2,3) vanish identically, i.e.,

$$S^3 = 0 \quad , \quad s^{3j} = 0 \tag{5.23}$$

We notice that the composite stress, the composite couple stress vector and the composite intrinsic force are to be specified by constitutive relations. Hence, there exist eighteen consitutive relations in the present theory.

The basic field equations for composite laminates follow from $(5.1)_a$ to $(5.1)_e$ and are given by

$$\begin{aligned}
&a: \quad \dot{\rho} + \frac{\dot{g}}{2g}\rho = 0 \\
&b: \quad \mathbf{T}^i{}_{,i} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \\
&c: \quad \mathbf{S}^i{}_{,i} + g^{1/2}(\rho\mathbf{c} - \mathbf{k}) = \rho g^{1/2}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \\
&d: \quad \mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,i} \times \mathbf{S}^i + g^{1/2}\mathbf{d} \times \mathbf{k} = 0 \\
&e: \quad \rho g^{1/2}\dot{\varepsilon} = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} = g^{1/2}P
\end{aligned} \tag{5.24}$$

where P is the mechanical power density (per element of volume) of the composite and is given by

$$g^{1/2}P = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} \tag{5.25}$$

The basic field equations (5.24) are both simple and elegant in form. In practice, we usually work with the components of the various fields. Hence, we now proceed to deduce the basic field equations in tensor components. We introduce the contravariant and covariant components of acceleration (α^i,α_i), director acceleration (β^i,β_i), body force (b^i,b_i), body couple (c^i,c_i), and

those of intrinsic force (k^i, k_i) as follows:

$$\dot{\mathbf{v}} = \alpha^i \mathbf{g}_i = \alpha_i \mathbf{g}^i \quad , \quad \dot{\mathbf{w}} = \beta^i \mathbf{g}_i = \beta_i \mathbf{g}^i$$
$$\mathbf{b} = b^i \mathbf{g}_i = b_i \mathbf{g}^i \quad , \quad \mathbf{c} = c^i \mathbf{g}_i = c_i \mathbf{g}^i \quad , \quad \mathbf{k} = k^i \mathbf{g}_i = k_i \mathbf{g}^i \tag{5.26}$$

The basic field equations (5.24) when expressed in component forms will reduce to

$$\text{a} : \quad \dot{\rho} + \frac{\dot{g}}{2g}\rho = 0$$
$$\text{b} : \quad \tau^{ij}{}_{\|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j)$$
$$\text{c} : \quad s^{ij}{}_{\|i} + (\rho c^j - k^j) = \rho(y^1\alpha^j + y^2\beta^j) \tag{5.27}$$
$$\text{d} : \quad \varepsilon_{ijn}(\tau^{ij} + d^i{}_{\|m}s^{mj} + d^i k^j) = \varepsilon_{ijn}(\tau^{ij} - s^{mi}\lambda^j{}_m - k^i d^j) = 0$$
$$\text{e} : \quad \rho\dot{\varepsilon} = \tau^{ij}v_{j\|i} + s^{ij}w_{j\|i} + k^i w_i = P$$

while the expression for mechanical power density P takes the form

$$\rho\dot{\varepsilon} = \tau^{ij}v_{j\|i} + s^{ij}w_{j\|i} + k^i w_i = P \tag{5.28}$$

With reference to $(5.27)_d$ we observe that the symmetry of the stress tensor is not valid. However, because ε_{ijk} is skew-symmetric with respect to i and j, it follows that the quantity in the parentheses in $(5.27)_d$ must be symmetric with respect to i and j. Hence the quantities

$$\tau'^{ij} = \tau^{ij} - s^{mi}\lambda^j{}_m - k^i d^j \tag{5.29}$$

$$\mathcal{T}^{ij} = \tau^{ij} + d^i{}_{\|m}s^{mj} + d^i k^j \tag{5.30}$$

are symmetric. We call τ'^{ij} and $\mathcal{T}^{ij}$ the *composite assigned symmetric tensors* or simply the *composite symmetric tensors.* As will become clear shortly, expression (5.29) is more convenient to work with. We notice that in the absence of the director, i.e.,

$$\mathbf{d} = 0 \quad \text{or} \quad d^i = 0$$

the composite symmetric tensor $\mathcal{T}^{ij}$ reduces to the classical symmetric tensor. It can be shown that in the absence of the micro-structure and the director the basic field equations (5.27) as well as the expressions for power reduce to those of classical continuum mechanics.

6. Constitutive equations for nonlinear elastic composite laminates.

This section presents an account of the development of constitutive relations for elastic composite laminates. While we confine our attention here to elastic composite laminates, it should be noted that the previous developments in sections 2 to 5 are not limited to elastic laminated composites.

Within the scope of the theory discussed in sections 2 to 5, we discuss the constitutive relations for elastic composite laminates in the presence of finite deformation and in the context of purely mechanical theory.

We recall that a material is defined by a *constitutive* assumption which characterizes the mechanical behavior of the medium. The constitutive assumption places a restriction on the processes which are admissible in a body — here the composite laminate.

We define an elastic composite laminate by a set of response functions which, in the context of purely mechanical theory, depend on appropriate kinematic variables. In our present discussion the set of response functions consists of the following functions

$$\mathbf{T}^i \ , \ \mathbf{S}^i \ , \ \mathbf{k} \tag{6.1}$$

or an equivalent set

$$\tau^{ij}(\text{or } \tau^{ij}) \ , \ s^{ij} \ , \ k^i \tag{6.2}$$

We introduce constitutive relations which must hold at each composite material point (macro particle) and for all time (t) in terms of the response functions (6.1). In this connection, we recall that the displacement function $\mathbf{r}$ in (3.1) is the place occupied by the *composite particle* P (with coordinates θ^i) in the present configuration, and the function $\mathbf{d}$ in (3.1) is the director, at the same composite material point, representing the effect of micro-structure. Thus the local state of an elastic composite laminate can be defined by functions $\mathbf{r}$ and $\mathbf{d}$ and their gradients at

each composite material point in the present configuration, namely

$$\mathbf{r} \;,\; \mathbf{r}_{,i} \;,\; \mathbf{d} \;,\; \mathbf{d}_{,i}$$

At this point, for convenience we recall the expression for mechanical power P, i.e.,

$$g^{1/2}P = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} \tag{6.3}$$

or equivalently

$$P = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^i w_i \tag{6.4}$$

We continue our discussion by assuming the existence of a **strain energy** or stored energy $\psi = \psi(\theta^i,t)$ per unit mass ρ of the composite laminate such that $\rho\dot{\psi}$ is equal to the mechanical power defined by (6.4), i.e.,

$$P = \rho\dot{\psi} \tag{6.5}$$

In the development of nonlinear constitutive equations for elastic composite lamiantes, we assume that the strain energy density ψ at each material point of the composite laminate (macro-particle) and for all t is specified by a response function which depends on **r**, **d** and their partial derivatives with respect to θ^i. Hence, the constitutive relation for the composite strain energy density may be stated as

$$\psi = \psi(\mathbf{r},\mathbf{r}_{,i},\mathbf{d},\mathbf{d}_{,i}\,;\,X) \tag{6.6}$$

Since the response function must remain unaltered under superposed rigid body translational displacement, the dependence on **r** must be excluded. In addition, we have already shown that $\mathbf{S}^3$ vanishes identically. Therefore, the constitutive assumption for the strain energy density of the composite laminate can now be written as

$$\psi = \bar{\psi}(\mathbf{r}_{,i},\mathbf{d},\mathbf{d}_{,\alpha}\,;\,X) \tag{6.7}$$

We also make similar constitutive assumptions for $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$. We make a note that in these constitutive equations, which represent the mechanical response of the medium, the dependence of the response function on the local geometrical properties of a reference state and material inhomogeneity is indicated through the argument X [1]. Here we limit the discussion to an elastic composite laminate which is homogeneous in its reference configuration and suppose that the dependence of the response functions on the properties of the reference state occurs through the values of the kinematical variables in the reference configuration. Therefore, in place of (6.7) we may write

$$\psi = \bar{\psi}(\mathbf{r}_{,i},\mathbf{d},\mathbf{d}_{,\alpha} ; \mathrm{R}_{,i},\mathbf{D},\mathbf{D}_{,\alpha}) \tag{6.8}$$

or

$$\psi = \bar{\psi}(\mathbf{g}_i,\mathbf{d},\mathbf{d}_{,i} ; \mathbf{G}_i,\mathbf{D},\mathbf{D}_{,i}) \tag{6.9}$$

Since

$$\mathbf{g}_i = \mathbf{r}_{,i} \quad , \quad \mathbf{G}_i = \mathbf{R}_{,i} \tag{6.10}$$

Following the same argument, we can arrive at similar constitutive assumptions for $\mathbf{T}^i$,$\mathbf{S}^i$ and $\mathbf{k}$. From (6.9) we obtain [2]

$$\dot{\psi} = \dot{\bar{\psi}} = \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i}\cdot\dot{\mathbf{g}_i} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}}\cdot\dot{\mathbf{d}} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}_{,i}}\cdot\dot{\mathbf{d}}_{,i} = \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i}\cdot\mathbf{v}_{,i} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}}\cdot\mathbf{w} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}_{,i}}\cdot\mathbf{w}_{,i} \tag{6.11}$$

Since

---

[1] See [Carroll and Naghdi, 1972].

[2] Operators of the form $\frac{\partial f}{\partial \mathbf{x}}$, where f is a scalar function of a vector variable occurring in (6.11) and elsewhere are defined as partial derivatives with respect to $\mathbf{x}$ satisfying

$$\lim_{\varepsilon\to 0}\frac{f(\mathbf{x}+\varepsilon\mathbf{v})-f(\mathbf{x})}{\varepsilon} = \frac{\partial f}{\partial \mathbf{x}}\cdot\mathbf{v}$$

for all values of the arbitrary vector $\mathbf{v}$.

$$\dot{\mathbf{g}}_i = (\frac{\partial \mathbf{p}}{\partial\theta^i})^{\cdot} = \frac{\partial}{\partial\theta^i}(\dot{\mathbf{p}}) = \frac{\partial \mathbf{v}}{\partial\theta^i} = \mathbf{v}_{,i} \tag{6.12}$$

and

$$\dot{\mathbf{d}} = \mathbf{w} \quad , \quad \dot{\mathbf{d}}_{,i} = \mathbf{w}_{,i} \tag{6.13}$$

Introducing (6.11) into $(5.24)_e$, we obtain

$$\rho g^{1/2} \{ \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i} \cdot \mathbf{v}_{,i} + \frac{\partial\bar{\psi}}{\partial\mathbf{d}} \cdot \mathbf{w} + \frac{\partial\bar{\psi}}{\mathbf{d}_{,i}} \cdot \mathbf{w}_{,i} \} = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w}$$

or

$$(\mathbf{T}^i - \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i}) \cdot \mathbf{v}_{,i} + (\mathbf{S}^i - \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{d}_i}) \cdot \mathbf{w}_{,i} + (g^{1/2}\mathbf{k} - \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{d}}) \cdot \mathbf{w} = 0 \tag{6.14}$$

This must hold for all arbitrary values of vectors $\mathbf{v}_{,i}$,$\mathbf{w}_{,i}$ and $\mathbf{w}$. Since the quantities in the parentheses are independent of $\mathbf{v}_{,i}$,,$\mathbf{w}_{,i}$ and $\mathbf{w}$, we conclude that

$$\begin{aligned} \mathbf{T}^i &= \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i} \\ \mathbf{S}^i &= \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{d}_{,i}} \\ g^{1/2}\mathbf{k} &= \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{d}} \end{aligned} \tag{6.15}$$

These are the nonlinear constitutive equations for $\mathbf{T}^i$,$\mathbf{S}^i$ and $\mathbf{k}$ along with the condition

$$\mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,i} \times \mathbf{S}^i + g^{1/2}\, \mathbf{d} \times \mathbf{k} = 0 \tag{6.16}$$

which is imposed by the principle of the moment of momentum of the composite laminate and must be satisfied by the response function $\bar{\psi}$.

As with the equations of motion, it is convenient in applications to specific problems to obtain alternative forms of the above constitutive equations expressed in component forms. To

obtain the appropriate constitutive equations for τ^{ij}, s^{ij} and k^i we proceed as follows.

Recall the formulas

$$\mathbf{r} = r^i \mathbf{g}_i = r_i \mathbf{g}^i \quad , \quad \mathbf{d} = d^i \mathbf{g}_i = d_i \mathbf{g}^i \tag{6.17}$$

$$\mathbf{g}_i = \mathbf{r}_{,i} = (r^m \mathbf{g}_m)_{,i} = r^m{}_{,i} \mathbf{g}_m + r^m \mathbf{g}_{m,i} = r^m{}_{,i} \mathbf{g}_m + r^m \{{}_i{}^n{}_m\} \mathbf{g}_n$$

$$= (r^m{}_{,i} + r^n \{{}_i{}^m{}_n\}) \mathbf{g}_m = r^m{}_{|i} \mathbf{g}_m = r_{m|i} \mathbf{g}^m \tag{6.18}$$

and

$$\mathbf{d}_{,i} = (d^m \mathbf{g}_m)_{,i} = d^m{}_{,i} \mathbf{g}_m + d^m \mathbf{g}_{m,i} = d^m{}_{,i} \mathbf{g}_m + r^n \{{}_i{}^m{}_n\} \mathbf{g}_m = d^m{}_{|i} \mathbf{g}_m = d_{m|i} \mathbf{g}^m \tag{6.19}$$

Substituting (6.17) to (6.19) into (6.9) and keeping in mind that ψ is a scalar valued function, we may rewrite (6.9) as

$$\psi = \tilde{\psi}(r_{m|i}, d_m, d_{m|\alpha} ; R_{m|i}, D_m, D_{m|\alpha}) \tag{6.20}$$

where $\tilde{\psi}$ is now a different function than $\bar{\psi}$. From (6.20) by differentiation we obtain

$$\dot{\psi} = \dot{\tilde{\psi}} = \frac{\partial \tilde{\psi}}{\partial r_{m|i}} \overline{(\dot{r}_{m|i})} + \frac{\partial \tilde{\psi}}{\partial d_m} \dot{d}_m + \frac{\partial \tilde{\psi}}{\partial d_{m|i}} \overline{(\dot{d}_{m|i})} = \frac{\partial \tilde{\psi}}{\partial r_{m|i}} v_{m|i} + \frac{\partial \tilde{\psi}}{\partial d_m} v_m + \frac{\partial \tilde{\psi}}{\partial d_{m|i}} w_{m|i} \tag{6.21}$$

Substituting (6.21) into $(22.46)_e$, we obtain

$$\rho \{ \frac{\partial \tilde{\psi}}{\partial r_{m|i}} v_{m|i} + \frac{\partial \tilde{\psi}}{\partial d_m} v_m + \frac{\partial \tilde{\psi}}{\partial d_{m|i}} w_{m|i} \} = \tau^{ij} v_{d|i} + s^{ij} w_{j|i} + k^i w_i$$

or

$$(\tau^{ij} - \rho \frac{\partial \tilde{\psi}}{\partial r_{j|i}} \cdot v_{j|i} + (s^{ij} - \rho \frac{\partial \tilde{\psi}}{\partial d_{j|i}}) w_{j|i} + (k^i - \rho \frac{\partial \tilde{\psi}}{\partial d_i}) w_i = 0 \tag{6.22}$$

This must hold for all arbitrary values of $v_{j|i}$, $w_{j|i}$ and d_i. Since the quantities in the parentheses are independent of $v_{j|i}$, $w_{j|i}$ and d_i, we conclude that

$$\tau^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial r_{j|i}}$$

$$s^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial d_{j|i}} \qquad (6.23)$$

$$k^{i} = \rho \frac{\partial \tilde{\psi}}{\partial d_{i}}$$

These are the component forms of the constitutive equations for τ^{ij}, s^{ij} and k^i along with the condition

$$\varepsilon_{ijn}\{\tau^{ij} + d^{i}{}_{|m}s^{mj} + d^{i}k^{j}\} = \varepsilon_{ijn}\{\tau^{ij} - s^{mi}\lambda^{j}{}_{m} - k^{i}d^{j}\} = 0 \qquad (6.24)$$

which is imposed by the principal of the moment of momentum of the composite laminate and must be satisfied by the response function $\tilde{\psi}$.

Before proceeding further, we obtain an alternative form of constitutive equations. To this end we consider the expression for mechanical power (6.3), i.e.,

$$g^{1/2}P = \mathbf{T}^{i} \cdot \mathbf{v}_{,i} + \mathbf{S}^{i} \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w}$$

and by taking advantage of the expressions (3.23), (3.26), (3.28) we write

$$\begin{aligned}
g^{1/2}P = {} & (g^{1/2}\tau^{ij}\mathbf{g}_j) \cdot [(\eta_{ki} + \omega_{ki})\mathbf{g}^k] + \\
& (g^{1/2}s^{ij}\mathbf{g}_j) \cdot [\dot{\lambda}_{ki}\mathbf{g}^k + \lambda^{l}{}_{i}(\omega_{kl} - \eta_{kl})\mathbf{g}^k] + \\
& (g^{1/2}k^{m}\mathbf{g}_m) \cdot [\dot{d}_k\mathbf{g}^k + d^{i}(\omega_{ki} - \eta_{ki})\mathbf{g}^k] \\
= {} & g^{1/2}\{\tau^{ij}\eta_{ki}(\mathbf{g}_j \cdot \mathbf{g}^k) + \tau^{ij}\omega_{ki}(\mathbf{g}_j \cdot \mathbf{g}^k) + \\
& s^{ij}\dot{\lambda}_{ki}(\mathbf{g}_j \cdot \mathbf{g}^k) + s^{ij}\lambda^{l}{}_{i}(\omega_{kl} - \eta_{kl})(\mathbf{g}_j \cdot \mathbf{g}^k) + \\
& k^{j}\dot{d}_k(\mathbf{g}_j \cdot \mathbf{g}^k) + k^{j}d^{i}(\omega_{ki} - \eta_{ki})(\mathbf{g}_j \cdot \mathbf{g}^k)\}
\end{aligned}$$

or

$$g^{1/2}P = g^{1/2}\delta^k{}_j\{\tau^{ij}\eta_{ki} + \tau^{ij}\omega_{ki} + s^{ij}\dot{\lambda}_{ki} + s^{ij}\lambda^l{}_i(\omega_{kl} + \eta_{kl}) + k^j\dot{d}_k + k^jd^i(\omega_{ki} - \eta_{ki})\}$$

or

$$P = (\tau^{ij} - s^{mi}\lambda^j{}_m - k^id^j)\eta_{ji} + s^{ij}\dot{\lambda}_{ji} + k^i\dot{d}_i + (\tau^{ij} + s^{mj}\lambda^i{}_m + k^jd^i)\omega_{ji} \tag{6.25}$$

The last term on the right hand side of (6.25) is a product of a symmetric and a skew-symmetric tensor component; hence it vanishes identically and we obtain

$$P = (\tau^{ij} - s^{mi}\lambda^j{}_m - k^id^j)\eta_{ji} + s^{ij}\dot{\lambda}_{ji} + k^i\dot{d}_i \tag{6.26}$$

We now recall the expression for the symmetric commposite stress tensor τ'^{ij} and substitute for τ^{ij} in (6.26) to obtain

$$P = \tau'^{ij}\eta_{ji} + s^{ij}\dot{\lambda}_{ji} + k^i\dot{d}_i \tag{6.27}$$

Recall the kinematical variables

$$\gamma_{ij} = \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) = \frac{1}{2}(g_{ij} - G_{ij})$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij} \tag{6.28}$$

$$\gamma_i = d_i - D_i$$

From (6.28) we obtain

$$\dot{\gamma}_{ij} = \frac{1}{2}\dot{g}_{ij} = \frac{1}{2}(2\eta_{ij}) = \eta_{ij}$$

$$\dot{\mathcal{K}}_{ij} = \dot{\lambda}_{ij} \tag{6.29}$$

$$\dot{\gamma}_i = \dot{d}_i$$

The expression of power (6.26) in terms of the kinematical variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i is

$$\rho\dot{\varepsilon} = \tau'^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ji} + k^i\dot{\gamma}_i = P \tag{6.30}$$

where

BASE

$$P = \mathcal{T}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ji} + k^{i}\dot{\gamma}_{i} \tag{6.31}$$

Rewriting the $\tilde{\psi}$ as a function of the variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i, i.e.,

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_i) \tag{6.32}$$

we obtain

$$\dot{\varepsilon} = \frac{\partial\psi}{\partial\gamma_{ij}}\dot{\gamma}_{ij} + \frac{\partial\psi}{\partial\mathcal{K}_{ij}}\dot{\mathcal{K}}_{ij} + \frac{\partial\psi}{\partial\gamma_i}\dot{\gamma}_i \tag{6.33}$$

From (6.30) and (6.32) we obtain

$$(\tau'^{ij} - \rho\frac{\partial\psi}{\partial\gamma_{ij}})\dot{\gamma}_{ij} + (s^{ij} - \rho\frac{\partial\psi}{\partial\mathcal{K}_{ji}})\dot{\mathcal{K}}_{ji} + (k^{i} - \rho\frac{\partial\psi}{\partial\gamma_i})\dot{\gamma}_i = 0 \tag{6.34}$$

Then by the usual procedure we obtain

$$\tau'^{ij} = \rho\frac{\partial\psi}{\partial\gamma_{ij}}$$

$$s^{ij} = \rho\frac{\partial\psi}{\partial\mathcal{K}_{ji}} \tag{6.35}$$

$$k^{i} = \rho\frac{\partial\psi}{\partial\gamma_i}$$

7. The complete theory

We recapitulate in this section the complete theory of elastic composite laminate in the context of purely mechanical theory.

The initial boundary value problem in the general theory.

The basic field equations of the nonlinear theory consist of the equations of motion and the energy equation given by (5.27) and repeated below for convenience:

$$
\begin{aligned}
&a: \quad \tau^{ij}{}_{|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j) \\
&b: \quad s^{ij}{}_{|i} + (\rho c^j - k^j) = \rho(y^1\alpha^j + y^2\beta^j) \\
&c: \quad \varepsilon_{ijn}(\tau^{ij} + \lambda^i{}_n s^{mj} + d^i k^j) = \varepsilon_{ijn}(\tau^{ij} - s^{mi}\lambda^j{}_m - k^i d^j) = 0 \\
&d: \quad \rho\dot{\varepsilon} = \tau^{ij} v_{j|i} + s^{ij} w_{j|i} + k^i w_i = P
\end{aligned}
\tag{7.1}
$$

where P is given by

$$P = \tau^{ij} v_{j|i} + s^{ij} w_{j|i} + k^i w_i \tag{7.2}$$

or equivalently by

$$P = \tau'^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i \tag{7.3}$$

where τ'^{ij} is the composite symmetric stress tensor and is given by

$$\tau'^{ij} = \tau^{ij} - s^{mi}\lambda^j{}_m - k^i d^j \tag{7.4}$$

The constitutive equations for an elastic composite laminate are specified by

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_i) \tag{7.5}$$

and

$$\tau'^{ij} = \rho \frac{\partial \psi}{\partial \gamma_{ij}} \tag{7.6}$$

$$s^{ij} = \rho \frac{\partial \psi}{\partial \mathcal{K}_{ji}} \tag{7.7}$$

$$k^{i} = \rho \frac{\partial \psi}{\partial \gamma_{i}} \tag{7.8}$$

We recall that (7.7) is subjected to the condition

$$s^{i3} = 0 \tag{7.9}$$

We note that instead of (7.5) to (7.8), any other alternative equivalent expressions derived in section 6 may be used.

The above field equations and constitutive relations characterize the initial boundary-value problem in the nonlinear theory of an elastic composite laminate.

The problem of establishing boundary conditions is not always clear in the literature on continuum theory of composites. Even in the case of mathematically coherent continuum theories with micro-structure the physical interpretations are not given or are ambiguous. Indeed, most (if not all) of the problems that are treated using various continuum theories for composites deal with periodic wave propagation or those problems for which the boundary conditions are not of primary importance.

The nature of the boundary conditions in the present theory may be seen from the rate of work expression for the composite contact force and the composite contact couple, i.e.,

$$\mathcal{R}_c = \int_{\partial \mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w}) da \tag{7.10}$$

The conditions at the boundary surface of the composite laminate at which the surface forces $\bar{\mathbf{t}}$ and the surface couples are prescribed require that

$$\mathbf{t} = \bar{\mathbf{t}} \quad , \quad \mathbf{s} = \bar{\mathbf{s}} \tag{7.11}$$

If we express the surface forces $\bar{\mathbf{t}}$ and the surface couples $\bar{\mathbf{s}}$ in terms of their components, i.e.,

$$\bar{\mathbf{t}} = \bar{t}^i \mathbf{g}_i = \bar{t}_i \mathbf{g}^i \tag{7.12}$$

$$\bar{\mathbf{s}} = \bar{s}^i \mathbf{g}_i = \bar{s}_i \mathbf{g}^i \tag{7.13}$$

and then using (5.21) and (5.22) the boundary conditions take the following forms:

$$\tau^{ij} n_i = \bar{t}^j \quad , \quad \tau^i{}_j n_i = \bar{t}_j \tag{7.14}$$

$$s^{ij} n_i = \bar{s}^j \quad , \quad s^i{}_j n_i = \bar{s}_j \tag{7.15}$$

To elaborate, we recall that our choice of convected coordinates is such that at a point P with coordinates θ^i $(i = 1,2,3)$ of the composite laminate, the coordinates θ^1, θ^2 are in fact the coordinate curves of the ply passing through the point P. Moreover, the coordinate θ^3 is in the direction of lay-up. This implies that for an arbitrary part $\mathcal{P}$, the boundary surface $\partial\mathcal{P}$ consists of two material surfaces of the form

$$\partial\mathcal{P}_1 \;:\; \theta^3 = \theta^3(\theta^\alpha) = C_1$$

and

$$\partial\mathcal{P}_1 \;:\; \theta^3 = \theta^3(\theta^\alpha) = C_2 \tag{7.16}$$

and a lateral material surface of the form

$$\partial\mathcal{P}_l \;:\; f(\theta^\alpha) = 0 \tag{7.17}$$

such that θ^3 = const. are closed smooth curves on the surface (7.17). With this background, it should now be clear that $\bar{t}^1, \bar{t}^2$ in $(7.14)_1$ are the stress resultants in $\mathbf{g}_1$, $\mathbf{g}_2$ directions, respectively, while $\bar{t}^3$ is the stress in $\mathbf{g}_3$ direction. Similarly, $\bar{s}^1, \bar{s}^2$ in $(7.15)_1$ are the stress couple resultants in $\mathbf{g}_1$, $\mathbf{g}_2$ directions, respectively, but $\bar{s}^3$ is the stress couple resultant in $\mathbf{g}_3$ direction, which is identically zero.

8. A constrained theory of composite laminates

The continuum theory presented in sections 2 to 7 is general and without any restriction/condition placed on the kinematical variables. Therefore the field equations and the constitutive relations are applicable to any elastic composite laminate. We now turn to the discussion of a constrained theory of our continuum model which may appropriately be called *Cosserat composite.*

We impose the condition that plies of the composite laminate do not separate from or slide over each other at all time during the motion of the composite laminate. This means the displacement vector of the material points throughout the body including at the interface must be single valued. Hence we obtain the following constraint conditions:

$$\mathbf{g}^{\alpha} \cdot \mathbf{d} = 0 \qquad (\alpha = 1,2) \tag{8.1}$$

Differentiating (8.1) with respect to time, we obtain

$$\dot{\mathbf{g}}^{\alpha} \cdot \mathbf{d} + \mathbf{g}^{\alpha} \cdot \mathbf{w} = 0 \qquad (\alpha = 1,2) \tag{8.2}$$

We recall

$$\mathbf{g}^{i} \cdot \mathbf{g}_{j} = \delta^{i}{}_{j} \tag{8.3}$$

Differentiating (8.3) with respect to time, we obtain

$$\dot{\mathbf{g}}^{i} \cdot \mathbf{g}_{j} = -\,\mathbf{g}^{i} \cdot \mathbf{v}_{,j} \tag{8.4}$$

and

$$\dot{\mathbf{g}}^{i} = -\,v^{i}{}_{|m}\mathbf{g}_{m} \tag{8.5}$$

Substituting (8.4), (8.5) into (8.2), we arrive at

$$d^i g^\alpha \cdot v_{,i} - g^\alpha \cdot w = 0 \tag{8.6}$$

and

$$d_m g^{im} g^{j\alpha} v_{j|i} - g^{i\alpha} w_i = 0 \qquad (\alpha = 1,2) \tag{8.7}$$

This is another form of the constraints (8.1) which is more appropriate for our present development.

For a composite laminate with constraints we assume that each of the functions T^i, S^i and k^i are determined to within an additive constraint response so that

$$\begin{aligned} T^i &= \bar{T}^i + \hat{T}^i \\ S^i &= \bar{S}^i + \hat{S}^i \\ k &= \bar{k} + \hat{k} \end{aligned} \tag{8.8}$$

where

$$\hat{T}^i \quad , \quad \hat{S}^i \quad , \quad \hat{k} \tag{8.9}$$

are specified by constitutive equations and

$$\bar{T}^i \quad , \quad \bar{S}^i \quad , \quad \bar{k} \tag{8.10}$$

which represent the response due to constraints (8.6) are arbitrary functions of θ^i,t, are workless and independent of the kinematical variables $v_{,i}$, $w_{,i}$ and w. Thus, recalling the expression (5.25) for mechanical power, we set

$$\bar{T}^i \cdot v_{,i} + \bar{S}^i \cdot w_{,i} + g^{1/2} \bar{k} \cdot w = 0 \tag{8.11}$$

This must hold for all values of the variables $v_{,i}$, $w_{,i}$ and w subject to the constraint condition (8.6). Multiplying (8.6) by the Lagrange multipliers δ_α $(\alpha = 1,2)$ and subtracting the results from (8.11), we obtain

$$(\tilde{\mathbf{T}}^i - \delta_\alpha d^i \mathbf{g}^\alpha) \cdot \mathbf{v}_{,i} + \tilde{\mathbf{S}}^i \cdot \mathbf{w}_{,i} + (g^{1/2}\tilde{\mathbf{k}} + \delta_\alpha \mathbf{g}^\alpha) \cdot \mathbf{w} = 0 \tag{8.12}$$

From (8.12) and the fact that $\tilde{\mathbf{T}}^i$, $\tilde{\mathbf{S}}^i$ and $\tilde{\mathbf{k}}$ are independent of $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$ it follows that

$$\tilde{\mathbf{T}}^i = \delta_\alpha d^i \mathbf{g}^\alpha \tag{8.13}$$

$$\tilde{\mathbf{S}}^i = 0 \tag{8.14}$$

$$g^{1/2}\tilde{\mathbf{k}} = -\,\delta_\alpha \mathbf{g}^\alpha \tag{8.15}$$

Expressions (8.13) to (8.15) represent the constraint response induced by the constraint equations (8.1). Substituting (8.13), (8.14) and (8.15) into linear momentum equation $(5.24)_b$ and the director momentum equation $(5.24)_c$, we obtain

$$[\hat{\mathbf{T}}^i + \delta_\alpha d^i \mathbf{g}^\alpha]_{,i} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) \tag{8.16}$$

and

$$\hat{\mathbf{S}}^i{}_{,i} + \rho g^{1/2}\mathbf{c} - [g^{1/2}\hat{\mathbf{k}} - \delta_\alpha \mathbf{g}^\alpha] = \rho g^{1/2}(y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) \tag{8.17}$$

Introducing the following temporary variables $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ by

$$\hat{\mathbf{b}} = \mathbf{b} - (\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}})$$

and

$$\hat{\mathbf{c}} = \mathbf{c} - (y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) \tag{8.18}$$

we can rewrite (8.16) and (8.17) as follows

$$\rho g^{1/2}\hat{\mathbf{b}} + \hat{\mathbf{T}}^i{}_{,i} + (\delta_\alpha d^i \mathbf{g}^\alpha)_{,i} = 0 \tag{8.19}$$

$$\rho g^{1/2}\hat{\mathbf{c}} + \hat{\mathbf{S}}^i{}_{,i} - g^{1/2}\hat{\mathbf{k}} + \delta_\alpha \mathbf{g}^\alpha = 0 \tag{8.20}$$

From (8.20) we obtain

$$(\delta_\alpha d^i \mathbf{g}^\alpha)_{,i} = -\,(\rho g^{1/2} d^i \hat{\mathbf{c}} + d^i \hat{\mathbf{S}}^j{}_{,j} - g^{1/2} d^i \hat{\mathbf{k}})_{,i} \tag{8.21}$$

Substitute (8.21) into (8.19) to obtain

$$\rho g^{1/2}\hat{\mathbf{b}} + \hat{\mathbf{T}}^i{}_{,i} - (\rho g^{1/2} d^i \hat{\mathbf{c}} + d^i \hat{\mathbf{S}}^j{}_{,j} - g^{1/2} d^i \mathbf{k})_{,i} = 0 \tag{8.22}$$

Moreover, from (8.20) and (8.1) we obtain

$$\rho g^{1/2}\mathbf{d} \cdot \hat{\mathbf{c}} + \mathbf{d} \cdot \hat{\mathbf{S}}^j{}_{,j} - g^{1/2}\mathbf{d} \cdot \hat{\mathbf{k}} = 0 \tag{8.23}$$

Recalling that $\hat{\mathbf{T}}^i$, $\hat{\mathbf{S}}^i$ and $\hat{\mathbf{k}}$ are specified as functions of various kinematical variables, it is clear that the system of equations (8.22) and (8.23) represent two equations for the determination of the primary unknowns $\mathbf{v}$ (or $\mathbf{r}$) and $\mathbf{d}$.

We now proceed to obtain the counterparts of (8.22) and (8.23) in component form. To this end, we assume, for an elastic composite laminate with constraint, the functions τ^{ij}, s^{ij} and k^i are determined to within an additive constraint response so that

$$\begin{aligned} \tau^{ij} &= \tilde{\tau}^{ij} + \hat{\tau}^{ij} \\ s^{ij} &= \tilde{s}^{ij} + \hat{s}^{ij} \\ k^i &= \tilde{k}^i + \hat{k}^i \end{aligned} \tag{8.24}$$

where

$$\hat{\tau}^{ij} \ , \ \hat{s}^{ij} \ , \ \hat{k}^i \tag{8.25}$$

are specified by constitutive equations and

$$\tilde{\tau}^{ij} \ , \ \tilde{s}^{ij} \ , \ \tilde{k}^i \tag{8.26}$$

which represent the response due to constraints (8.7), are arbitrary functions of θ^i,t, workless and independent of kinematical variables $v_{i|j}$, $w_{i|j}$ and w_i. Thus, recalling the expression (5.28) for mechanical power, we set

$$\tilde{\tau}^{ij} v_{j|i} + \tilde{s}^{ij} w_{j|i} + \tilde{k}^i w_i = 0 \tag{8.27}$$

which must hold for all values of the variables $v_{j|i}$, $w_{j|i}$ and w_i subject to the constraint conditions (8.7). Multiplying (8.7) by the Lagrange multipliers λ_α ($\alpha = 1,2$) and subtracting the results from (8.27), we obtain [3]

$$(\tilde{\tau}^{ij} - \lambda_\alpha d^i g^{j\alpha}) v_{j\,|\,i} + \tilde{s}^{ij} w_{j\,|\,i} + (\tilde{k}^i + \lambda_\alpha g^{i\alpha}) w_i = 0 \tag{8.28}$$

From (8.28) and the fact that $\tilde{\tau}^{ij}$, $\tilde{s}^{ij}$ and $\tilde{k}$ are independent of $v_{j|i}$, $w_{j|i}$ and w_i it follows that

$$\tilde{\tau}^{ij} = \lambda_\alpha d^i g^{j\alpha} \tag{8.29}$$

$$\tilde{s}^{ij} = 0 \tag{8.30}$$

$$\tilde{k}^i = -\lambda_\alpha g^{i\alpha} \tag{8.31}$$

Substituting (8.29), (8.30) and (8.31) into $(5.27)_b$ and $(5.27)_c$, we obtain

$$[\hat{\tau}^{ij} + \lambda_\gamma d^i g^{j\gamma}]_{\,|\,i} + \rho b^j = \rho(\alpha^j + y^1 \beta^j) \tag{8.32}$$

and

$$\hat{s}^{ij}{}_{|\,i} - [\hat{k}^j - \lambda_\gamma g^{j\gamma}] + \rho c^j = \rho(y^1 \alpha^j + y^1 \beta^j) \tag{8.33}$$

From (8.18) we have

$$\hat{b}^j = b^j - (\alpha^j + y^1 \beta^j)$$

and (8.34)

$$\hat{c}^j = c^j - (y^1 \alpha^j + y^2 \beta^j)$$

Making use of (8.3) we rewrite (8.32) and (8.33) as follows

$$\rho \hat{b}^j + \hat{\tau}^{ij}{}_{|\,i} + (\lambda_\gamma d^i g^{j\gamma})_{|\,i} = 0 \tag{8.35}$$

and

---

[3] Note that λ_α is now different from δ_α.

$$\rho\hat{c}^j + \hat{s}^{ij}{}_{|i} - \hat{k}^j + \lambda_\gamma g^{\gamma j} = 0 \tag{8.36}$$

From (8.36) we obtain

$$(\lambda_\gamma d^i g^{\gamma j})_{|i} = -(\rho d^i \hat{c}^j + d^i \hat{s}^{mj}{}_{|m} - d^i \hat{k}^j)_{|i} \tag{8.37}$$

and substitute into (8.35) to obtain

$$\rho\hat{b}^j + \hat{\tau}^{ij}{}_{|i} - (\rho d^i \hat{c}^j + \rho d^i \hat{s}^{mj}{}_{|m} - d^i \hat{k}^j)_{|i} = 0 \tag{8.38}$$

Moreover, from (8.36) we obtain

$$\rho d_j \hat{c}^j + d_j \hat{s}^{ij}{}_{|i} - d_j \hat{k}^j + \lambda_\alpha d_j g^{\beta j} = 0 \tag{8.39}$$

However, from (8.1) we have

$$d_j g^{\beta j} = 0 \tag{8.40}$$

Hence, by (8.39) and (8.40) we have

$$\rho d_j \hat{c}^j + d_j \hat{s}^{ij}{}_{|i} - d_j \hat{k}^j = 0 \tag{8.41}$$

Again recalling that $\hat{\tau}^{ij}$, $\hat{s}^{ij}$ and $\hat{k}^i$ are specified, by constitutive equations, as functions of relevant kinematical variables, it is clear that the system of equations (8.38) and (8.41) represent four equations for the determination of four primary unknowns v_i and d.

Before closing this section, we obtain a relation between the Lagrange multipliers δ_α and λ_α. Recalling ( ), we may write (8.13) as follows

$$\tilde{T}^i = g^{1/2}\tilde{\tau}^{ij} g_j = \delta_\alpha d^i g^{j\alpha} g_j$$

or

$$g^{1/2}(\tilde{\tau}^{ij} - g^{-1/2}\delta_\alpha d^i g^{j\alpha}) g_j = 0 \tag{8.42}$$

Since $g^{1/2} \neq 0$ and g_j are linearly independent base vectors, we obtain

$$\tilde{\tau}^{ij} = g^{-1/2}\delta_\alpha d^i g^{j\alpha} \tag{8.43}$$

A comparison between (8.43) and (8.29) yields

$$\lambda_\alpha = g^{-1/2}\delta_\alpha \tag{8.44}$$

Similarly, from (8.15) we obtain

$$g^{1/2}\tilde{k}^i g_i = - \delta_\alpha g^{i\alpha} g_i$$

or

$$g^{1/2}(\tilde{k}^i + g^{-1/2}\delta_\alpha g^{i\alpha} g_i) = 0 \tag{8.45}$$

Again, since g_i are independent base vectors and since $g^{1/2} \neq 0$, we obtain

$$\tilde{k}^i = - g^{-1/2}\delta_\alpha g^{i\alpha} \tag{8.46}$$

Comparing (8.46) and (8.31), we obtain

$$\lambda_\alpha = g^{-1/2}\delta_\alpha \tag{8.47}$$

which confirms the results (8.44).

9. Linearized kinematics

This section is devoted to the linearized form of the kinematical results of section (3). In particular, we deduce the linearized kinematic measures of a composite laminate with infinitesimal displacements and infinitesimal director displacements as a special case of the general results in section (3). We note that a bold vertical bar, in the linearized expressions, will denote covariant differentiation with respect to G_{ij} corresponding to the reference configuration.

We recall the expressions

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,\theta^3,t) + \xi\mathbf{d}(\eta^\alpha,\theta^3,t) \tag{9.1}$$

and

$$\mathbf{P}^* = \mathbf{R}(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3) \tag{9.2}$$

Within the context of linear theory of composite continuum we let [1]

$$\mathbf{p}^* = \mathbf{P}^* + \varepsilon\mathbf{u}^* \tag{9.3}$$

where ε is a non-dimensional parameter and $\mathbf{u}^*$ is a three-dimensional vector such that

$$\mathbf{u}^* = u^{*i}\mathbf{g}_i = u_i^*\mathbf{g}^i \tag{9.4}$$

$$\mathbf{u}^* = \mathbf{u}^*(\eta^\alpha,\xi,\theta^3,t) = \mathbf{u}(\eta^\alpha,\theta^3,t) + \xi\delta(\eta^\alpha,\theta^3,t) \tag{9.5}$$

From (9.3) we obtain

$$\mathbf{v}^* = \varepsilon\dot{\mathbf{u}}^* \tag{9.6}$$

Introducing (9.5) into (9.3) and making use of (9.2), we obtain

[1] The use of ε in this section is temporary, clear from the context and not to be confused with the use of the same notation in the previous section.

$$\mathbf{p}^* = [\mathbf{R}(\eta^\alpha,\theta^3) + \varepsilon\mathbf{u}(\eta^\alpha,\theta^3,t)] + \xi[\mathbf{D}(\eta^\alpha,\theta^3) + \varepsilon\boldsymbol{\delta}(\eta^\alpha,t)] \tag{9.7}$$

By a comparison between (9.1) and (9.7) we conclude that

$$\begin{aligned} \mathbf{r}(\theta^i,t) &= \mathbf{R}(\theta^i) + \varepsilon\mathbf{u}(\theta^i,t) \\ \mathbf{d}(\theta^i,t) &= \mathbf{D}(\theta^i) + \varepsilon\boldsymbol{\delta}(\theta^i,t) \end{aligned} \tag{9.8}$$

where we have identified η^α with θ^α. The velocity and the director velocity are readily obtained as

$$\begin{aligned} \mathbf{v} &= \varepsilon\dot{\mathbf{u}} \\ \mathbf{w} &= \varepsilon\dot{\boldsymbol{\delta}} \end{aligned} \tag{9.9}$$

We say that the motion of a laminated composite continuum characterized by (9.8) describes infinitesimal deformation if the magnitudes of $\mathbf{u}$, $\boldsymbol{\delta}$ and all their derivatives are bounded and are of the same order as $\mathbf{R}$ and $\mathbf{D}$ and if

$$\varepsilon \ll 1 \tag{9.10}$$

In what follows we shall be concerned with (scalar, vector or tensor) functions of position and time, determined by $\varepsilon\mathbf{u}$ and $\varepsilon\boldsymbol{\delta}$ and their space and time derivatives. We denote these functions by the customary order symbol $O(\varepsilon^n)$ if there exists a real number C, independent of $\varepsilon,\mathbf{u},\boldsymbol{\delta}$ and their derivatives such that

$$|\,O(\varepsilon^n)\,| < C\,\varepsilon^n \qquad \varepsilon \to 0 \tag{9.11}$$

We would like to emphasize that the infinitesimal theory which we wish to obtain as a special case of the results in section (3) and in the sense of (9.10) is such that all kinematical quantities (including the displacement $\mathbf{u}$, the director displacement $\boldsymbol{\delta}$ and other kinematical measures, as well as their space and time derivatives are all of $O(\varepsilon)$.

The base vectors $\mathbf{g}_i^*$ can be obtained from (9.7):

$$\mathbf{g}_\alpha^* = \mathbf{p}_{,\alpha}^* = (\mathbf{R}_{,\alpha} + \varepsilon\mathbf{u}_{,\alpha}) + \xi(\mathbf{D}_{,\alpha} + \varepsilon\boldsymbol{\delta}_{,\alpha})$$
$$\mathbf{g}_3^* = \mathbf{p}_3^* = \mathbf{D} + \varepsilon\boldsymbol{\delta} \tag{9.12}$$

Similarly the base vectors $\mathbf{g}_i$ are obtained from $(9.8)_1$

$$\mathbf{g}_i = \mathbf{r}_{,i} = (\mathbf{R} + \varepsilon\mathbf{u})_{,i} = \mathbf{G}_{,i} + \varepsilon\mathbf{u}_{,i} \tag{9.13}$$

We now proceed to obtain the relative kinematic measures γ_{ij}, $\mathcal{K}_{ij}$ and γ_i. To this end we first obtain

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = (\mathbf{G}_i + \varepsilon\mathbf{u}_i) \cdot (\mathbf{G}_j + \varepsilon\mathbf{u}_j) = G_{ij} + \varepsilon(\mathbf{G}_i \cdot \mathbf{u}_j + \mathbf{G}_{,j} \cdot \mathbf{u}_{,i}) + O(\varepsilon^2) \tag{9.14}$$

$$d_i = \mathbf{g}_i \cdot \mathbf{d} = (\mathbf{G}_i + \varepsilon\mathbf{u}_{,i}) \cdot (\mathbf{D} + \varepsilon\boldsymbol{\delta}) = D_i + \varepsilon(\mathbf{G}_i \cdot \boldsymbol{\delta} + \mathbf{u}_{,i} \cdot \mathbf{D}) + O(\varepsilon^2) \tag{9.15}$$

$$\lambda_{ij} = \mathbf{g}_i \cdot \mathbf{d}_{,j} = (\mathbf{G}_i + \varepsilon\mathbf{u}_{,i}) \cdot (\mathbf{D}_{,j} + \varepsilon\boldsymbol{\delta}_{,j}) = \Lambda_{ij} + \varepsilon(\mathbf{G}_i \cdot \boldsymbol{\delta}_{,j} + \mathbf{u}_{,i} \cdot \mathbf{D}_{,j}) + O(\varepsilon^2) \tag{9.16}$$

From (9.14) to (9.16) we obtain

$$\gamma_{ij} = \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) = \frac{1}{2}\varepsilon(\mathbf{G}_i \cdot \mathbf{u}_{,j} + \mathbf{G}_j \cdot \mathbf{u}_{,i}) + O(\varepsilon^2) \tag{9.17}$$

$$\gamma_i = d_i - D_i = \varepsilon(\mathbf{G}_i \cdot \boldsymbol{\delta} + \mathbf{u}_{,i} \cdot \mathbf{D}) + O(\varepsilon^2) \tag{9.18}$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij} = \varepsilon(\mathbf{G}_i \cdot \boldsymbol{\delta}_{,j} + \mathbf{u}_{,i} \cdot \mathbf{D}_{,j}) + O(\varepsilon^2) \tag{9.19}$$

At this stage it is desirable to elaborate on the manner in which the process of linearization may be carried out. To this end we take $\mathbf{u}'$ and $\boldsymbol{\delta}'$ to be vector functions defined by

$$\mathbf{u}' = \varepsilon\mathbf{u} = O(\varepsilon) \qquad u'^i = \mathbf{A}^i \cdot \mathbf{u}' = O(\varepsilon) \tag{9.20}$$

$$\boldsymbol{\delta}' = \varepsilon\boldsymbol{\delta} = O(\varepsilon) \qquad \delta'^i = \mathbf{A}^i \cdot \boldsymbol{\delta}' = O(\varepsilon) \tag{9.21}$$

Making use of (9.20) and (9.21) we may rewrite (9.17) to (9.19) as

$$\gamma'_{ij} = \frac{1}{2}(\mathbf{G}_i \cdot \mathbf{u}'_{,j} + \mathbf{G}_j \cdot \mathbf{u}'_{,i}) \tag{9.22}$$

$$\gamma'_i = (\mathbf{G}_i \cdot \delta' + \mathbf{u}'_{,i} \cdot \mathbf{D}) \tag{9.23}$$

$$\mathcal{K}'_{ij} = (\mathbf{G}_i \cdot \delta'_{,j} + \mathbf{u}'_{,i} \cdot \mathbf{D}_{,j}) \tag{9.24}$$

where we have introduced γ'_{ij}, γ'_i $\mathcal{K}'_{ij}$ which are of $O(\varepsilon)$ we have

$$\gamma_{ij} = \gamma'_{ij} + O(\varepsilon^2) = O(\varepsilon) \tag{9.25}$$

$$\gamma_i = \gamma'_i + O(\varepsilon^2) = O(\varepsilon) \tag{9.26}$$

$$\mathcal{K}_{ij} = \mathcal{K}'_{ij} + O(\varepsilon^2) = O(\varepsilon) \tag{9.27}$$

We also have

$$\left(\frac{g}{G}\right)^{1/2} = 1 + \gamma'^{i}_{i} + O(\varepsilon^2) \tag{9.28}$$

and

$$\frac{g}{G} = 1 + 2g^{ij}\gamma'_{ij} + O(\varepsilon^2) \tag{9.29}$$

We now retain only terms of $O(\varepsilon)$ in expressions such as (9.25) and hence approximate γ_{ij}, γ_i and $\mathcal{K}_{ij}$ by γ'_{ij}, γ'_i and $\mathcal{K}_{ij}$, etc. In order to avoid the introduction of unnecessary additional notations we proceed with linearization by retaining only terms of $O(\varepsilon)$ and after the approximation without loss of generality, we set $\varepsilon = 1$. In this manner the relative kinematic measures γ_{ij}, γ_i and $\mathcal{K}_{ij}$ reduce to

$$\gamma_{ij} = \frac{1}{2}(\mathbf{G}_i \cdot \mathbf{u}_{,j} + \mathbf{G}_j \cdot \mathbf{u}_{,i}) \tag{9.30}$$

$$\gamma_i = \mathbf{G}_i \cdot \delta + \mathbf{u}_{,i} \cdot \mathbf{D} \tag{9.31}$$

$$\mathcal{K}_{ij} = \mathbf{G}_i \cdot \delta_{,j} + \mathbf{u}_{,i} \cdot \mathbf{D}_{,j} \tag{9.32}$$

We also obtain

$$\rho g^{1/2} = \rho_o G^{1/2}$$

or

$$\rho G^{1/2}(1 + \gamma^i_i) = \rho_o G^{1/2}$$

or

$$\rho = \rho_o \frac{1}{1 + \gamma^i_i} = \rho_o(1 - \gamma^i_i) \tag{9.33}$$

10. Linearized field equations

Previously, with reference to the linearization of the kinematical results for a composite, it was assumed that all kinematic measures such as γ_{ij}, γ_i and $\mathcal{K}_{ij}$ as well as their space and time derivatives are of $O(\varepsilon)$. These must now be supplemented by additional assumptions in a complete infinitesimal theory. We now assume that the vector fields $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$ are all zero in the reference configuration. We further assume that $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$ (or their components) when expressed in suitable non-dimensional forms, as well as their space and time derivatives are all of $O(\varepsilon)$.

Recalling the linearization procedure of the previous section and avoiding the introduction of additional notations, we now regard $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$ as infinitesimal quantities of $O(\varepsilon)$. As a result of linearization, all tensor quantities are now referred to the initial undeformed surface and covariant differentiation is with respect to G_{ij} in the reference configuration. It then follows that in the equations $(7.1)_{a,b,c,}$ *each term is of* $O(\varepsilon)$ *and that* d^i, *and* $d^i{}_{|m} = \lambda^i{}_m$ *or* λ_{im} *must* be replaced to the order of ε by D^i and $\Lambda^i{}_m$ or Λ_{im}, respectively. We omit the details since it is a straightforward calculation and merely record the linearized version of the equations of motion as follows.

$$\tau^i{}_{j|i} + \rho_o b_j = \rho_o(\ddot{u}_j + y^1\ddot{\delta}_j) \tag{10.1}$$

$$s^i{}_{j|i} + (\rho_o c_j - k_j) = \rho_o(y^1\ddot{u}_j + y^2\ddot{\delta}_j) \tag{10.2}$$

$$\varepsilon_{ijn}\{\tau^{ij} + s^{mj}\Lambda^i{}_m + D^i k^j\} = \varepsilon_{ijn}\{\tau^{ij} - s^{mi}\Lambda^j{}_m - k^i D^j] = 0 \tag{10.3}$$

where the vertical bar in (10.1) to (10.3) and the rest of this section denotes covariant differentiation with respect to G_{ij}. We also note that all quantities are now referred to the base vectors $\mathbf{G}_i$ of the reference configuration.

Moreover, upon linearization we obtain

$$\tau'^{ij} = \tau^{ij} - s^{mi}\Lambda^j{}_m - k^i D^j \tag{10.4}$$

In the light of the assumptions stated above and expression (10.4), the energy equation takes the form

$$\rho_o\dot{\varepsilon} = \tau'^{ij}\dot{\gamma}_{ji} + s^{ij}\dot{\mathcal{K}}_{ji} + k^{i}\dot{\gamma}_{i} = P \tag{10.5}$$

11. Preliminaries to linearized constitutive equation

Before proceeding further, we dispose of some results which are independent of linearization; however, they will be particular useful in the applications of the linear theory. First, we recall that the position vector $\mathbf{P}^*$, of the micro-body $\mathcal{B}^*$, in the reference configuration is given by

$$\mathbf{P}^* = \mathbf{R}(\eta^\alpha,\theta^3) + \xi\, \mathbf{D}(\eta^\alpha,\theta^3) \tag{11.1}$$

In general, $\mathbf{D}$ in (11.1) is a three-dimensional vector having components D^1,D^2,D^3 in the directions of $\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3$. However, in the reference configuration without loss of generality we may specify $\mathbf{D}$ by

$$\mathbf{D} = D\mathbf{A}_3 \quad , \quad D_\alpha = 0 \quad , \quad D_3 = D(\eta^\alpha,\theta^3) \tag{11.2}$$

where $\mathbf{A}_3 = \mathbf{A}_3\,(\eta^\alpha)$ is the unit normal to the Cosserat surface, i.e., the shell-like representative element at composite particle P. From (11.1) and (11.2) it follows that the base vectors $\mathbf{G}_i^*$ and the metric tensor G_{ij}^*, of the micro-structure, in the reference (initial) configuration are

$$\mathbf{G}_\alpha^* = \mathbf{R}_{,\alpha} + \xi\mathbf{D}_{,\alpha} = \mathbf{G}_\alpha + \xi(D\mathbf{A}_3)_{,\alpha} = \mathbf{G}_\alpha + \xi(D_{,\alpha}\mathbf{A}_3 + D\mathbf{A}_{3,\alpha}) = \mathbf{G}_\alpha + \xi D\mathbf{A}_{3,\alpha} + \xi D_{,\alpha}\mathbf{A}_3 \tag{11.3}$$

and

$$\mathbf{G}_3^* = \mathbf{D} = D\mathbf{A}_3 \tag{11.4}$$

We recall the results

$$\mathbf{A}_3 \cdot \mathbf{A}_\beta = 0 \;\Rightarrow\; \mathbf{A}_{3,\alpha} \cdot \mathbf{A}_\beta + \mathbf{A}_3 \cdot \mathbf{A}_{\beta,\alpha} = 0$$

Hence,

$$\mathbf{A}_\beta \cdot \mathbf{A}_{3,\alpha} = -\,\mathbf{A}_3 \cdot \mathbf{A}_{\beta,\alpha} = -\,B_{\beta\alpha} = -\,B_{\alpha\beta} \tag{11.5}$$

and

$$A_{3,\alpha} = -B^{\beta}_{\alpha}A_{\beta} \tag{11.6}$$

where $B_{\alpha\beta}$ are the components of the second fundamental form of the surface. By (11.3), (11.6) and the fact that $A_\alpha = G_\alpha$, we obtain

$$G^*_\alpha = G_\alpha - \xi D B^\beta{}_\alpha A_\beta + \xi D_{,\alpha} A_3 = G_\beta \delta^\beta{}_\alpha - \xi D B^\beta{}_\alpha G_\beta + \xi D_{,\alpha} A_3$$

$$= (\delta^\beta{}_\alpha - \xi D B^\beta{}_\alpha) G_\beta + \xi D_{,\alpha} A_3 \tag{11.7}$$

Let

$$\nu^\beta{}_\alpha = \delta^\beta{}_\alpha - \xi D B^\beta{}_\alpha \tag{11.8}$$

Then by (11.4), (11.7) and (11.8) we have

$$\begin{aligned} G^*_\alpha &= \nu^\beta{}_\alpha G_\beta + \xi D_{,\alpha} A_3 \\ G^*_3 &= D A_3 \end{aligned} \tag{11.9}$$

and hence,

$$\begin{aligned} G^*_{\alpha\beta} &= \nu^\gamma{}_\alpha \nu^\delta{}_\beta G_{\gamma\delta} + \xi^2 D_{,\alpha} D_{,\beta} \\ G^*_{\alpha 3} &= \xi D D_{,\alpha} = \frac{1}{2}\,\xi (D^2)_{,\alpha} \\ G^*_{33} &= D^2 \end{aligned} \tag{11.10}$$

Let us now introduce a set of curvilinear coordinates ζ^i such that $\zeta^\alpha = \eta^\alpha$ and where ζ^3 is measured to the scale of the rectangular Cartesian coordinates (say $x^i = x_i$) along the positive direction of the uniquely defined normal A_3 of the Cosserat surface (i.e., micro-structure). Now in the reference configuration, which we take to be the initial configuration, the convected general curvilinear coordinates θ^i can always be related to ζ^i with ζ^3 as a specified function of η^α and ξ. For the purpose of this investigation and to avoid unnecessary complications, we denote ζ^3 simply by ζ and specify it by

$$\zeta = \zeta(\eta^\alpha)\xi \tag{11.11}$$

where ζ is a function of η^α only. In the special case that $\zeta(\eta^\alpha) = 1$ we obtain $\zeta = \xi$ in the

reference configuration. The coordinate system $\{\eta^\alpha,\xi\}$ where ξ is measured along the normal to the Cosserat surface is called normal coordinate system. Thus with ζ specified by (11.11), the position vector $\mathbf{P}^*$ of the micro-body $\mathcal{B}^*$ in the reference configuration referred to the normal coordinates is given by

$$\mathbf{P}^* = \mathbf{R}(\eta^\alpha,\theta^3) + \zeta\mathbf{A}_3(\eta^\alpha,\theta^3) \tag{11.12}$$

Let $\overline{\mathbf{G}}_i^*$ and $\overline{\mathbf{G}}_{ij}^*$ denote the base vectors and the metric tensor associated with the normal coordinates. From (11.12) we obtain

$$\overline{\mathbf{G}}_\alpha^* = \mathbf{R}_{,\alpha} + \zeta\mathbf{A}_{3,\alpha} = \mathbf{G}_\beta\delta^\beta{}_\alpha - \zeta B^\beta{}_\alpha\mathbf{A}_\beta = (\delta^\beta{}_\alpha - \zeta B^\beta{}_\alpha)\mathbf{G}_\beta$$

Hence, we have

$$\begin{aligned} \overline{\mathbf{G}}_\alpha^* &= \mu^\beta{}_\alpha\mathbf{G}_\beta \\ \overline{\mathbf{G}}_3^* &= \mathbf{A}_3 \end{aligned} \tag{11.13}$$

where

$$\mu^\beta{}_\alpha = \delta^\beta{}_\alpha - \zeta B^\beta{}_\alpha \tag{11.14}$$

From (11.13) we have:

$$\begin{aligned} \overline{G}_{\alpha\beta}^* &= \mu^\gamma{}_\alpha\mu^\delta{}_\beta G_{\gamma\delta} \\ \overline{G}_{\alpha 3}^* &= 0 \\ \overline{G}_{33}^* &= 1 \end{aligned} \tag{11.15}$$

A comparison between (11.1) and (11.2) with $\mathbf{D}$ specified by (11.2) reveals that

$$\zeta = D\xi \tag{11.16}$$

which is the transformation relation between ζ and ξ. Moreover, under this transformation, we obtain from (11.8), (11.14) and (11.15)

BASE

$$\nu^{\beta}{}_{\alpha} = \delta^{\beta}{}_{\alpha} - \xi D B^{\beta}{}_{\alpha} = \delta^{\beta}{}_{\alpha} - \zeta B^{\beta}{}_{\alpha} = \mu^{\beta}{}_{\alpha} \tag{11.17}$$

If we let det $(\nu^{\beta}{}_{\alpha}) = \frac{\nu}{D}$ and det $(\mu^{\beta}{}_{\alpha}) = \mu$ we obtain

$$\mu = \frac{\nu}{D} \tag{11.18}$$

It is worth noting that the metric tensors G^*_{ij} and $\overline{G}^*_{ij}$ become identical when evaluated on the surface $\xi = 0$ or $\zeta = 0$ in the reference configuration and are both given by

$$\begin{aligned} G^*_{\alpha\beta} &= \overline{G}^*_{\alpha\beta} = G_{\alpha\beta} \\ G^*_{\alpha 3} &= \overline{G}^*_{\alpha 3} = 0 \\ G^*_{33} &= \overline{G}^*_{33} = 1 \end{aligned} \tag{11.19}$$

We now proceed to obtain expressions for G_i, G_{ij} and $G^{1/2}$ corresponding to coordinates θ^i. Consistent with the kinematic assumption (11.2) we take the function $\mathbf{R}(\theta^{\alpha},\theta^3)$ to be

$$\mathbf{R}(\theta^{\alpha},\theta^3) = \overline{\mathbf{R}}(\theta^{\alpha}) + \theta^3 \mathbf{A}_3(\theta^{\alpha}) \tag{11.20}$$

From this we have

$$\begin{aligned} \mathbf{G}_{\alpha} &= \mathbf{R}_{,\alpha} = \overline{\mathbf{R}}_{,\alpha} + \theta^3 \mathbf{A}_{3,\alpha} = \overline{\mathbf{R}}_{,\alpha} - \theta^3 B^{\gamma}{}_{\alpha} \mathbf{A}_{\gamma} \\ \mathbf{G}_3 &= \mathbf{R}_{,3} = (\theta^3 \mathbf{A}_3)_{,3} = \mathbf{A}_3 \end{aligned} \tag{11.21}$$

and

$$\begin{aligned} G_{\alpha\beta} &= (\overline{\mathbf{R}}_{,\alpha} - \theta^3 B^{\gamma}{}_{\alpha}\mathbf{A}_{\gamma}) \cdot (\overline{\mathbf{R}}_{,\beta} - \theta^3 B^{\delta}{}_{\beta}\mathbf{A}_{\delta}) \\ &= \overline{\mathbf{R}}_{,\alpha} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 B^{\gamma}{}_{\alpha}\mathbf{A}_{\gamma} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 B^{\delta}{}_{\beta}\mathbf{A}_{\delta} \cdot \overline{\mathbf{R}}_{,\alpha} + (\theta^3)^2 B^{\gamma}{}_{\alpha} B^{\delta}{}_{\beta} \mathbf{A}_{\gamma} \cdot \mathbf{A}_{\delta} \\ &= \overline{\mathbf{R}}_{,\alpha} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 (B^{\gamma}{}_{\alpha}\mathbf{A}_{\gamma} \cdot \overline{\mathbf{R}}_{,\beta} - B^{\delta}{}_{\beta}\mathbf{A}_{\gamma} \cdot \overline{\mathbf{R}}_{,\alpha}) + (\theta^3)^2 B^{\gamma}{}_{\alpha} B^{\delta}{}_{\beta} \mathbf{A}_{\gamma} \cdot \mathbf{A}_{\delta} \\ &= \overline{\mathbf{R}}_{,\alpha} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 (B^{\gamma}{}_{\alpha}\overline{\mathbf{R}}_{,\beta} + B^{\gamma}{}_{\beta}\overline{\mathbf{R}}_{,\alpha}) \cdot \mathbf{A}_{\gamma} + (\theta^3)^2 B^{\gamma}{}_{\alpha} B^{\delta}{}_{\beta} \mathbf{A}_{\gamma} \cdot \mathbf{A}_{\delta} \end{aligned}$$

Hence we have

$$G_{\alpha\beta} = \bar{\mathbf{R}}_{,\alpha} \cdot \bar{\mathbf{R}}_{,\beta} - \theta^3(B^\gamma{}_\alpha \bar{\mathbf{R}}_{,\beta} + B^\gamma{}_\beta \bar{\mathbf{R}}_{,\alpha}) \cdot \mathbf{A}_\gamma + (\theta^3)^2 B^\gamma{}_\alpha B^\delta{}_\beta A_{\gamma\delta}$$

$$G_{\alpha 3} = (\bar{\mathbf{R}}_{,\alpha} - \theta^3 B^\gamma{}_\alpha \mathbf{A}_\gamma) \cdot \mathbf{A}_3 = 0 \tag{11.22}$$

$$G_{33} = \mathbf{A}_3 \cdot \mathbf{A}_3 = 1$$

Also,

$$G^{1/2} = [\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3] = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3 = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{A}_3$$

$$= [(\bar{\mathbf{R}}_{,1} - \theta^3 B^\gamma{}_1 \mathbf{A}_\gamma) \times (\bar{\mathbf{R}}_{,2} - \theta^3 B^\delta{}_2 \mathbf{A}_\delta)] \cdot \mathbf{A}_3$$

$$= (\bar{\mathbf{R}}_{,1} \times \bar{\mathbf{R}}_{,2}) \cdot \mathbf{A}_3 - \theta^3[B^\gamma{}_1(\mathbf{A}_\gamma \times \bar{\mathbf{R}}_{,2}) + B^\gamma{}_2(\bar{\mathbf{R}}_{,1} \times \mathbf{A}_\gamma)] \cdot \mathbf{A}_3$$

$$+ (\theta^3)^2 B^\gamma{}_1 B^\delta{}_2 (\mathbf{A}_\gamma \times \mathbf{A}_\delta) \cdot \mathbf{A}_3 \tag{11.23}$$

We now combine the assumptions (11.2) and (11.20) to obtain from (11.1)

$$\mathbf{P}^*(\theta^\alpha, \theta^3, \xi) = \mathbf{R} + \xi D \mathbf{A}_3 = \bar{\mathbf{R}} + \theta^3 \mathbf{A}_3 + \xi D \mathbf{A}_3 \tag{11.24}$$

From this we obtain

$$\mathbf{G}^*_\alpha = \mathbf{R}_{,\alpha} + \xi(D\mathbf{A}_3)_{,\alpha} = \mathbf{R}_{,\alpha} + \xi D_{,\alpha} \mathbf{A}_3 - \xi D B^\gamma{}_\alpha \mathbf{A}_\gamma$$

Hence, we have

$$\mathbf{G}^*_\alpha = \mathbf{G}_\alpha - \xi D B^\gamma{}_\alpha \mathbf{G}_\gamma + \xi D_{,\alpha} \mathbf{A}_3 = v^\gamma{}_\alpha \mathbf{G}_\gamma + \xi D_{,\alpha} \mathbf{A}_3$$

$$\tag{11.25}$$

$$\mathbf{G}^*_3 = D\mathbf{A}_3$$

where

$$v^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \xi D B^\gamma{}_\alpha \tag{11.26}$$

Moreover, from (11.25) we obtain

BASE

$$G^*_{\alpha\beta} = (\nu^\gamma{}_\alpha G_\gamma + \xi D_{,\alpha} A_3) \cdot (\nu^\delta{}_\beta G_\delta + \xi D_{,\beta} A_3) = \nu^\gamma{}_\alpha \nu^\delta{}_\gamma G_{\gamma\delta} + \xi^2 D_{,\alpha} D_{,\beta}$$

$$G^*_{\alpha 3} = (\nu^\gamma{}_\alpha G_\gamma + \xi D_{,\alpha} A_3) \cdot (D A_3) = \xi D D_{,\alpha} = \frac{1}{2}\xi (D^2)_{,\alpha} \tag{11.27}$$

$$G^*_{33} = (D A_3) \cdot (D A_3) = D^2$$

and

$$G^{*1/2} = [G^*_1, G^*_2, G^*_3] = (G^*_1 \times G^*_1) \cdot G^*_3$$

$$= \{(\nu^\gamma{}_1 G_\gamma + \xi D_{,1} A_3) \times (\nu^\gamma{}_2 G_\delta + \xi D_{,2} A_3)\} \cdot (D A_3)$$

$$= \{\nu^\gamma{}_1 \nu^\gamma{}_2 (G_\gamma \times G_\delta) + \xi D_{,2} \nu^\gamma{}_1 (G_\gamma \times A_3) + \xi D_{,1} \nu^\delta{}_2 (A_3 \times G_\delta)\} \cdot (D A_3)$$

$$= D \nu^\gamma{}_1 \nu^\delta{}_2 (G_\gamma \times G_\delta) \cdot A_3$$

$$= D\{\nu^1{}_1 \nu^2{}_2 (G_1 \times G_2) + \nu^2{}_1 \nu^1{}_2 (G_2 \times G_1)\} \cdot A_3$$

$$= D\{\nu^1{}_1 \nu^2{}_2 (G_1 \times G_2) + \nu^2{}_1 \nu^1{}_2 (G_2 \times G_1)\} \cdot A_3$$

$$= D(\nu^1{}_1 \nu^2{}_2 - \nu^1{}_2 \nu^2{}_1)(G_1 \times G_2) \cdot G_3 = D G^{1/2} \det(\nu^\alpha{}_\beta) \tag{11.28}$$

where in obtaining (11.28) we have made use of (11.23). Since

$$\nu = D \det(\nu^\beta{}_\alpha)$$

we obtain from (11.28)

$$\nu = D \det(\nu^\beta{}_\alpha) = \left(\frac{G^*}{G}\right)^{1/2} \tag{11.29}$$

In the rest of this development we assume each ply of the composite is sufficiently thin and confine our attention to the field equations of the linearized theory (10.1) and (10.2). Moreover, for the position vector **R** and the director **D**, in the reference (initial) configuration, we adopt the assumptions (11.20) and (11.2). Hence, in the reference configuration we have

$$\mathbf{R}(\theta^\alpha,\theta^3) = \bar{\mathbf{R}}(\theta^\alpha) + \theta^3\mathbf{A}_3 \tag{11.30}$$

$$\mathbf{D} = D\mathbf{A}_3 \quad , \quad D_\alpha = 0 \quad , \quad D_3 = D(\eta^\alpha) = D(\theta^\alpha) \tag{11.31}$$

and

$$\mathbf{P}^* = \mathbf{R} + \xi D\mathbf{A}_3 = \bar{\mathbf{R}} + (\theta^3 + \xi D)\mathbf{A}_3 \tag{11.32}$$

As mentioned before, within the scope of the linear theory $\mathbf{g}_i, \mathbf{g}_i^*, \mathbf{a}_i, g^{1/2}, g^{*1/2}$ and $a^{1/2}$ may be replaced by their reference (initial) values in the definitions of the various resultants. We now proceed to obtain the resultants which occur in the linearized equations of motion. Consider $\mathbf{T}^\alpha$ and within the context of the linear theory make use of (5.7), (5.21) to write

$$\mathbf{T}^\alpha = G^{1/2}\tau^{\alpha j}\mathbf{G}_j = \frac{1}{\xi_2}\int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi = \frac{1}{\xi_2}\int_0^{\xi_2} G^{*1/2}\tau^{*\alpha j}\mathbf{G}_j^* d\xi$$

or

$$\begin{aligned} G^{1/2}(\tau^{\alpha\beta}\mathbf{G}_\beta + \tau^{\alpha 3}\mathbf{G}_3) &= \frac{1}{\xi_2}\int_0^{\xi_2} G^{*1/2}(\tau^{*\alpha\beta}\mathbf{G}_\beta^* + \tau^{*\alpha 3}\mathbf{G}_3^*)d\xi \\ &= \frac{1}{\xi_2}\int_0^{\xi_2} G^{*1/2}\{\tau^{*\alpha\beta}(\nu\gamma_\beta\mathbf{G}_\gamma + \xi D_{,\beta}\mathbf{A}_3) + \tau^{*\alpha 3}D\mathbf{A}_3\}d\xi \\ &= \frac{1}{\xi_2}\int_0^{\xi_2} \nu G^{1/2}\tau^{*\alpha\beta}\nu\gamma_\beta\mathbf{G}_\gamma d\xi \\ &\quad + \frac{1}{\xi_2}\int_0^{\xi_2} \nu G^{1/2}(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha 3})\mathbf{A}_3 d\xi \end{aligned} \tag{11.33}$$

where in obtaining (11.33) we have made use of (11.25) and (11.29). Since $\mathbf{G}_\beta$ and $\mathbf{A}_3$ are linearly independent vectors and since $\mathbf{G}_\beta$, $\mathbf{A}_3$ and $G^{1/2}$ are independent of ξ, it follows from (11.33) that

$$\tau^{\alpha\beta} = \frac{1}{\xi_2}\int_0^{\xi_2} \nu\tau^{*\alpha\gamma}\nu^\beta{}_\gamma d\xi \quad , \quad \tau^{\alpha 3} = \frac{1}{\xi_2}\int_0^{\xi_2} \nu(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha 3})d\xi \tag{11.34}$$

We note that the composite stress vector $\mathbf{T}^3$ is not related to $\mathbf{T}^{*3}$ (within each constituent of the composite) and must be specified by a constitutive relation separately. In a similar manner, we consider $\mathbf{S}^\alpha$ and within the context of linear theory we use (5.10), (5.22) to write

$$\mathbf{S}^\alpha = G^{1/2} s^{\alpha j}\mathbf{G}_j = \frac{1}{\xi_2}\int_0^{\xi_2} \mathbf{T}^{*\alpha}\xi \, d\xi = \frac{1}{\xi_2}\int_0^{\xi_2} G^{*1/2}\tau^{*\alpha j}\mathbf{G}^*{}_j \xi \, d\xi$$

or

$$G^{1/2}(s^{\alpha\beta}\mathbf{G}_\beta + s^{\alpha 3}\mathbf{G}_3) = \frac{1}{\xi_2}\int_0^{\xi_2} G^{*1/2}(\tau^{*\alpha\beta}\mathbf{G}^*_\beta + \tau^{*\alpha 3}\mathbf{G}^*_3)\xi \, d\xi$$

$$= \frac{1}{\xi_2}\int_0^{\xi_2} G^{*1/2}\{\tau^{*\alpha\beta}(\nu\gamma_\beta\mathbf{G}_\gamma + \xi D_{,\beta}\mathbf{A}_3) + \tau^{*\alpha 3}D\mathbf{A}_3\}\xi \, d\xi$$

$$= \frac{1}{\xi_2}\int_0^{\xi_2} \nu G^{1/2}\tau^{*\alpha\beta}\nu\gamma_\beta\mathbf{G}_\gamma\xi \, d\xi$$

$$+ \frac{1}{\xi_2}\int_0^{\xi_2} \nu G^{1/2}(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha 3})\mathbf{A}_3\xi \, d\xi \qquad (11.35)$$

where in obtaining (11.35) we have made use of (11.25) and (11.29). Since $\mathbf{G}_\beta$ and $\mathbf{A}_3$ are linearly independent and since $\mathbf{G}_\beta$, $\mathbf{A}_3$ and $G^{1/2}$ are not functions of ξ, we obtain from (11.6)

$$s^{\alpha\beta} = \frac{1}{\xi_2}\int_0^{\xi_2} \nu\tau^{*\alpha\gamma}\nu^\beta{}_\gamma\xi \, d\xi \quad , \quad s^{\alpha 3} = \frac{1}{\xi_2}\int_0^{\xi_2} \nu(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha 3})\xi \, d\xi \qquad (11.36)$$

We recall that

$$s^{3i} = 0 \qquad (11.37)$$

Next, we consider $\mathbf{k}$ and in the same manner we write

$$G^{1/2}\mathbf{k} = G^{1/2}(k^\alpha\mathbf{G}_\alpha + k^3\mathbf{G}_3) = \frac{1}{\xi_2}\int_0^{\xi_2} \mathbf{T}^{*3}d\xi = \frac{1}{\xi_2}\int_0^{\xi_2} G^{*1/2}\tau^{*3j}\mathbf{G}_j d\xi$$

or

$$G^{1/2}(k^\alpha G_\alpha + k^3 A_3) = \frac{1}{\xi_2}\int_o^{\xi_2} G^{*1/2}(\tau^{*3\alpha} G^*_\alpha + \tau^{*33} G^*_3) d\xi$$

$$= \frac{1}{\xi_2}\int_o^{\xi_2} G^{*1/2}\{\tau^{*3\alpha}(\nu^\gamma{}_\alpha G_\gamma + \xi D_{,\ldots}) + \tau^{*33} D A_3\} d\xi$$

$$= \frac{1}{\xi_2}\int_o^{\xi_2} \nu G^{1/2}\tau^{*3\alpha}\nu^\gamma{}_\alpha G_\gamma d\xi$$

$$+ \frac{1}{\xi_2}\int_o^{\xi_2} \nu G^{1/2}(\xi D_{,\alpha}\tau^{*3\alpha} + D\tau^{*33}) A_3 d\xi \qquad (11.38)$$

where again in obtaining (11.38) we have made use of (11.25) and (11.29). By the usual argument it follows from (11.38)

$$k^\alpha = \frac{1}{\xi_2}\int_o^{\xi_2} \nu\tau^{*3\gamma}\nu^\alpha{}_\gamma d\xi \quad , \quad k^3 = \frac{1}{\xi_2}\int_o^{\xi_2} \nu(\xi D_{,\alpha}\tau^{*3\alpha} + D\tau^{*33}) d\xi \qquad (11.39)$$

Collecting the results of this section, we have

$$\tau^{\alpha\beta} = \frac{1}{\xi_2}\int_o^{\xi_2} \nu\tau^{*\alpha\gamma}\nu^\beta{}_\gamma d\xi \quad , \quad \tau^{\alpha 3} = \frac{1}{\xi_2}\int_o^{\xi_2} \nu(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha\beta}) d\xi$$

τ^{3i} or T^3 must be specified directly by a constitutive equation.

$$s^{\alpha\beta} = \frac{1}{\xi_2}\int_o^{\xi_2} \nu\tau^{*\alpha\gamma}\nu^\beta{}_\gamma \xi \, d\xi \quad , \quad s^{\alpha 3} = \frac{1}{\xi_2}\int_o^{\xi_2} \nu(\xi D_{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha\beta})\xi \, d\xi$$

$$s^{3i} = 0 \quad \text{or} \quad S^3 = 0 \qquad (11.40)$$

$$k^\alpha = \frac{1}{\xi_2}\int_o^{\xi_2} \nu\tau^{*3\gamma}\nu^\alpha{}_\gamma d\xi \quad , \quad k^3 = \frac{1}{\xi_2}\int_o^{\xi_2} \nu(\xi D_{,\alpha}\tau^{*3\alpha} + D\tau^{*33}) d\xi$$

The resultants in (11.40) are defined in terms of the stress tensor τ^{*ij} referred to the convected coordinates $\eta^i = \{\theta^\alpha, \xi\}$.

Next, we proceed to obtain the counterparts of (11.40) in terms of normal coordiantes $\zeta^i = \{\zeta^\alpha, \zeta^3\} = \{\eta^\alpha, \zeta\} = \{\theta^\alpha, \zeta\}$ where we have

$$\zeta = D\xi \tag{11.41}$$

Let the contravariant stress tensor in the context of classical continuum mechanics, referred to the normal coordinates ζ^i be denoted by $\bar{\tau}^{*ij}$. The relationship between $\bar{\tau}^{*ij}$ and τ^{*ij} is obtained by making use of the transformation law between two second order tensors as follows:

$$\partial\eta^l \, \tau^{*kl} \tag{11.42}$$

Hence,

$$\bar{\tau}^{*\alpha\beta} = \frac{\partial\zeta^\alpha}{\partial\eta^k}\frac{\partial\zeta^\beta}{\partial\eta^l}\tau^{*kl} = \frac{\partial\zeta^\alpha}{\partial\eta^k}\left(\frac{\partial\zeta^\beta}{\partial\eta^\lambda}\tau^{*k\lambda} + \frac{\partial\zeta^\beta}{\partial\eta^3}\tau^{*k3}\right)$$

$$= \frac{\partial\zeta^\alpha}{\partial\eta^\gamma}\frac{\partial\zeta^\beta}{\partial\eta^\lambda}\tau^{*\gamma\lambda} + \frac{\partial\zeta^\alpha}{\partial\eta^3}\frac{\partial\zeta^\beta}{\partial\eta^\lambda}\tau^{*3\lambda}$$

$$= \delta^\alpha{}_\gamma\delta^\beta{}_\lambda\tau^{*\gamma\lambda} = \tau^{*\alpha\beta} \tag{11.43}$$

and

$$\bar{\tau}^{*\alpha 3} = \frac{\partial\zeta^\alpha}{\partial\eta^k}\frac{\partial\zeta^3}{\partial\eta^l}\tau^{*kl} = \frac{\partial\zeta^\alpha}{\partial\eta^k}\left(\frac{\partial\zeta^3}{\partial\eta^\lambda}\tau^{*k\lambda} + \frac{\partial\zeta^3}{\partial\xi}\tau^{*k3}\right)$$

$$= \frac{\partial\zeta^\alpha}{\partial\eta^\gamma}\frac{\partial\zeta^3}{\partial\eta^\lambda}\tau^{*\gamma\lambda} + \frac{\partial\zeta^\alpha}{\partial\eta^3}\frac{\partial\zeta^3}{\partial\eta^\lambda}\tau^{*3\lambda} + \frac{\partial\zeta^\alpha}{\partial\eta^\gamma}\frac{\partial\zeta^3}{\partial\xi}\tau^{*\gamma 3} + \frac{\partial\zeta^\alpha}{\partial\xi}\frac{\partial\zeta^3}{\partial\xi}\tau^{*33}$$

$$= \delta^\alpha{}_\gamma(\xi D_{,\lambda})\tau^{*\gamma\lambda} + D\delta^\alpha{}_\gamma\tau^{*\gamma 3} = \xi\, D^{,\beta}\tau^{*\alpha\beta} + D\tau^{*\alpha 3} \tag{11.44}$$

$$\bar{\tau}^{*33} = \frac{\partial\zeta^3}{\partial\eta^k}\frac{\partial\zeta^3}{\partial\eta^l}\tau^{*kl} = \frac{\partial\zeta^3}{\partial\eta^\gamma}\frac{\partial\zeta^3}{\partial\eta^l}\tau^{*\gamma l} + \frac{\partial\zeta^3}{\partial\xi}\frac{\partial\zeta^3}{\partial\eta^l}\tau^{*3l}$$

$$= \xi\, D_{,\gamma}\left(\frac{\partial\zeta^3}{\partial\eta^\lambda}\tau^{*\gamma\lambda} + \frac{\partial\zeta^3}{\partial\xi}\tau^{*\gamma 3}\right) + D\left(\frac{\partial\zeta^3}{\partial\eta^\lambda}\tau^{*3\lambda} + \frac{\partial\zeta^3}{\partial\xi}\tau^{*33}\right)$$

$$= \xi\, D_{,\gamma}(\xi\, D_{,\lambda}\tau^{*\gamma\lambda} + D\tau^{*\gamma 3}) + D(\xi\, D_{,\lambda}\tau^{*3\lambda} + D\tau^{*33})$$

$$= \xi^2 D_{,\alpha}D_{,\beta}\tau^{*\alpha\beta} + 2\xi D D_{,\alpha}\tau^{*\alpha 3} + D^2\tau^{*33} \tag{11.45}$$

We note that if the thickness of the representative element in the direction of normal is h_2, we

have

$$\zeta = 0 \quad \text{at} \quad \xi = 0$$
$$\zeta = h_2 \quad \text{at} \quad \xi = \xi_2 \tag{11.46}$$

Hence

$$D\xi_2 = h_2 \tag{11.47}$$

which relates ξ_2 to h_2 and D. In particular, if D = 1 we obtain $\mathbf{D} = A_3$ and

$$\xi_2 = h_2 \tag{11.48}$$

We now define a new set of composite field quantities in terms of $\bar{\tau}^{*ij}$ as follows

$$\bar{\tau}^{\alpha\beta} = \frac{1}{h_2}\int_o^{h_2} \mu\bar{\tau}^{*\alpha\gamma}\mu_\gamma^\beta d\xi \quad , \quad V^\alpha = \bar{\tau}^{\alpha 3} = \frac{1}{h_2}\int_o^{h_2} \mu\bar{\tau}^{*\alpha 3} d\xi$$

$$\bar{\tau}^{3\alpha} = \theta^3 D_{,\beta}\tau^{\beta\alpha} + D\tau^{3\alpha}$$

$$\bar{\tau}^{33} = (\theta^3)^2 D_{,\alpha}D_{,\beta}\tau^{\alpha\beta} + \theta^3 DD_{,\alpha}(\tau^{\alpha 3} + \tau^{3\alpha}) + D^2\tau^{33} \tag{11.49}$$

$$\bar{s}^{\alpha\beta} = \frac{1}{h_2}\int_o^{h_2} \mu\bar{\tau}^{*\alpha\gamma}\mu_\gamma^\beta \xi \, d\xi \quad , \quad \bar{s}^{\alpha 3} = \frac{1}{h_2}\int_o^{h_2} \mu\bar{\tau}^{*\alpha 3}\xi \, d\xi$$

$$\bar{s}^{3i} = 0$$

$$V^3 = \frac{1}{h_2}\int_o^{h_2} \mu(\bar{\tau}^{*33} - B_{\alpha\beta}\bar{\tau}^{*\alpha\gamma}\mu^\beta_\gamma\xi)d\xi$$

where μ^α_γ and μ are given previously by (11.17) and (11.18). Making use of (11.43) to (11.45) in (11.49) we obtain

$$\bar{\tau}^{\alpha\beta} = \tau^{\alpha\beta} \quad , \quad V^\alpha = \bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = Dk^\alpha + DB^\alpha_\gamma s^{\gamma 3} + s^{\beta\alpha}D_{,\beta}$$
$$\bar{s}^{ij} = Ds^{ij} \quad , \quad V^3 = Dk^3 - DB_{\alpha\beta}s^{\alpha\beta} + D_{,\alpha}s^{\alpha 3} \tag{11.50}$$

which relates the two sets of definitions (11.40) and (11.49).

12. Linear constitutive relations for elastic composite laminates

This section is concerned with the derivation of the constitutive relations for a composite laminate in terms of those of its constituents. In what follows we assume that each of the constituents of the laminated composites is a homogeneous isotropic elastic material. We recall that within the scope of the linear theory all kinematical variables are referred to the reference configuration. Previously we showed that the strain energy function, ψ may be written as

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_i) \tag{12.1}$$

We assume that in the case of the linear theory ψ is given by a quadratic function of the infinitesimal kinematical variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i. We also recall that after systematic linearization of the expression for power, we obtained for the linear theory

$$\rho_o \dot{\varepsilon} = \bar{\tau}^{ij}\dot{\gamma}_{ji} + s^{ij}\dot{\mathcal{K}}_{ji} + k^i\dot{\gamma}_i = P \tag{12.2}$$

Since the rates $\dot{\gamma}_{ij}$, $\dot{\mathcal{K}}_{ij}$ and $\dot{\gamma}_i$ are all independent and since the coefficients are rate independent, after substituting (12.2) into (12.3) we obtain

$$\rho_o \{ \frac{\partial \psi}{\partial \gamma_{ji}} \dot{\gamma}_{ji} + \frac{\partial \psi}{\partial \mathcal{K}_{ji}} \dot{\mathcal{K}}_{ji} + \frac{\partial \psi}{\partial \gamma_i} \dot{\gamma}_i \} = \bar{\tau}^{ij}\dot{\gamma}_{ji} + s^{ij}\dot{\mathcal{K}}_{ji} + k^i\dot{\gamma}_i \tag{12.3}$$

or

$$(\bar{\tau}^{ij} - \rho_o \frac{\partial \psi}{\partial \gamma_{ji}})\dot{\gamma}_{ji} + (s^{ij} - \rho_o \frac{\partial \psi}{\partial \mathcal{K}_{ji}})\dot{\mathcal{K}}_{ji} + (k^i - \rho_o \frac{\partial \psi}{\partial \gamma_i})\dot{\gamma}_i = 0 \tag{12.4}$$

Hence

$$\bar{\tau}^{ij} = \rho_o \frac{\partial \psi}{\partial \gamma_{ij}}$$

$$s^{ij} = \rho_o \frac{\partial \psi}{\partial \mathcal{K}_{ji}} \tag{12.5}$$

$$k^i = \rho_o \frac{\partial \psi}{\partial \gamma_i}$$

The relationship between the strain energy function ψ, per unit mass of the composite, and those of the constituents is given by

$$\psi = \frac{1}{\rho_o G^{1/2} \xi_2} \int_o^{\xi_2} \rho_o^* G^{*1/2} \psi^* d\xi \tag{12.6}$$

or

$$\rho_o \psi = \frac{1}{\xi_2} \int_o^{\xi_2} \nu(\rho_o^* \psi^*) d\xi = \frac{1}{h_2} \int_o^{h_2} \mu(\rho_o^* \psi^*) d\xi = \frac{1}{h_2} \int_o^{h_1} \mu(\rho_{o1}^* \psi_1^*) d\xi + \frac{1}{h_2} \int_{h_1}^{h_2} \mu(\rho_{o2}^* \psi_2^*) d\xi \tag{12.7}$$

where ρ_{o1}. and ρ_{o2}^* denote mass densities of the constituents $\hat{\mathcal{B}}_1^*$ and $\hat{\mathcal{B}}_2^*$. We recall that in three-dimensional linear theory we have

$$\rho_o^* \psi^* = \frac{1}{2} E_{mn}^{*ij} \gamma_{ij}^* \gamma^{*mn} \tag{12.8}$$

and

$$\tau^{*ij} = E_{mn}^{*ij} \gamma^{*mn} \tag{12.9}$$

We also recall that for isotropic elastic materials we have

$$E_{mn}^{*ij} = \lambda^* G^{*ij} G_{mn}^* + \mu^* (\delta^i{}_m \delta^j{}_n + \delta^i{}_n \delta^j{}_m) \tag{12.10}$$

$$\tau^{*ij} = \mu^* (G^{*im} G^{*jn} + G^{*in} G^{*jm} + \frac{2\nu^*}{1-2\nu^*} G^{*ij} G^{*mn}) \gamma_{mn}^* \tag{12.11}$$

$$\lambda^* = \frac{2\nu^*}{1-2\nu^*} \mu^* \tag{12.12}$$

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For an explicit set of constitutive relations the integration on the right hand side of (12.7) must be carried out using (12.8) for $\rho_o^* \psi^*$. Here we remark that as in the case of two dimensional theories of continuum mechanics (such as plates and shells), except possibly in very special cases, it appears to be extremely difficult to calculate the function ψ in (12.2) from the strain energy function ψ^* of the classical three dimensional theory. In the case of composite materials this becomes more complicated due to the existence of two (or more) materials.

Alternatively, in order to provide constitutive relations in which the coefficients are related to elastic constants of the constituents we can make use of the so-called specific Gibbs energy function. This method proves to be more convenient for the derivation of the linear constitutive equations for a composite laminate and will be described in the next section.

13. Linear constitutive relations for composite laminates: An alternative procedure

In this section we introduce an alternative procedure for the derivation of the linear constitutive equations for a composite laminate. The method takes advantage of the specific Gibbs energy function[2].

We recall that the central idea in the derivation of the constitutive relation for an elastic composite laminate was that the specific internal energy is given by a function of the form (12.1) where in the case of the linear theory it reduces to a quadratic function of its arguments. As mentioned previously, although expression (12.6) is elegant, the explicit integration of (12.6) in most cases becomes exceedingly difficult. Here we provide and alternative approach for explicit derivation of the constitutive relations (for the linear theory of a composite laminate) in which the coefficients are related to the elastic constants of the constituents.

We recall that the constitutive equations of the classical linear theory of elasticity in the context of purely mechanical theory may be expressed in terms of the three-dimensional specific Gibbs free energy function, say ϕ^*, in the form[3]

$$\gamma_{ij}^* = -\rho_o^* \frac{\partial \phi^*}{\partial \tau^{*ij}} \tag{13.1}$$

where γ_{ij}^* is the infinitesimal strain and where ϕ^* and ψ^* are related through

$$\phi^* = \phi^*(\tau^{*ij}) = \psi^*(\gamma_{ij}^*) - \frac{1}{\rho_o^*} \tau^{*ij}\gamma_{ij}^* \tag{13.2}$$

and ϕ^* and ψ^* are quadratic functions of their arguments and both also depend on the reference

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[2] This idea was first introduced by Green, Naghdi and Wenner [1971], in the context of Cosserat surface theory.

[3] The partial derivative $\frac{\partial \phi^*}{\partial \tau^{*ij}}$ is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial \phi^*}{\partial \tau^{*ij}} + \frac{\partial \phi^*}{\partial \tau^{*ij}}\right)$$

values of G^*_{ij}. It may be noted that the function ϕ^* defined by (13.2) is the negative of the expression for the complimentary energy density. We now recall that the Gibbs function ϕ^* for an initially homogeneous and isotropic material can be expressed as

$$\rho^*_o\phi^* = \{-\frac{1+\nu^*}{2E^*}G^*_{im}G^*_{jn} + \frac{\nu^*}{2E^*}G^*_{ij}G^*_{mn}\}\,\tau^{*ij}\tau^{*mn} \tag{13.3}$$

where G^*_{ij} is the initial metric tensor, E^* is Young's modulus of elasticity and ν^* is Poisson's ratio.

Within the scope of the linear theory and corresponding to (12.6) we define a *composite Gibbs free energy* (or a "composite complementary energy") ϕ as follows:

$$\rho_o G^{1/2}\phi = \frac{1}{\xi_2}\int_o^{\xi_2}\rho^*_o G^{*1/2}\phi^* d\xi \tag{13.4}$$

From (13.2), by integration with respect to ξ between zero and ξ^2 we obtain

$$\frac{1}{\xi_2}\int_o^{\xi_2}\rho^*_o G^{*1/2}\phi^* d\xi = \frac{1}{\xi_2}\int_o^{\xi_2}\rho^*_o G^{*1/2}\psi^* d\xi - \frac{1}{\xi_2}\int_o^{\xi_2}G^{*1/2}\tau^{*ij}\gamma^*_{ij}d\xi \tag{13.5}$$

Considering (12.6), (13.2) and (13.4), we may rewrite (13.5) as

$$\rho_o G^{1/2}\phi = \rho_o G^{1/2}\psi - \frac{1}{\xi_2}\int_o^{\xi_2}G^{*1/2}\tau^{*ij}\gamma^*_{ij}d\xi$$

or

$$\phi = \psi - \frac{1}{\rho_o\xi_2}\int_o^{\xi_2}\nu\tau^{*ij}\gamma^*_{ij}d\xi \tag{13.6}$$

where in obtaining (13.6) we have made use of (11.29). By making use of the expressions for τ^{*ij}, γ^*_{ij}, the expressions for various resultants and the kinematic assumptions for **R** and **D**, we can express the integral in (13.6) in terms of the various resultants and their corresponding relative kinematic measures. However, as before the constitutive relations for the interlaminar stress

vectors $\mathbf{T}^i$ *should be specified directly.* Keeping this and expressions (13.2) and (13.6) in mind, we assume the existence of a Gibbs free energy function ϕ, such that

$$\rho_o\phi = \rho_o\bar{\phi}(\bar{\tau}^{ij}, s^{ij}, k^i) = \rho_o\psi - \{\bar{\tau}^{ij}\gamma_{ij} + s^{ij}\mathcal{K}_{ij} + k^i\gamma_i\} \tag{13.7}$$

Differentiating both sides of (13.7) with respect to t, we obtain

$$\rho_o\dot{\mathcal{E}} = \rho_o\dot{\psi} = \rho_o\dot{\bar{\phi}} + \dot{\overline{(\bar{\tau}^{ij}\gamma_{ij})}} + \dot{\overline{(s^{ij}\mathcal{K}_{ij})}} + \dot{\overline{(k^i\gamma_i)}}$$

$$= \rho_o\dot{\bar{\phi}} + \dot{\bar{\tau}}^{ij}\gamma_{ij} + \bar{\tau}^{ij}\dot{\gamma}_{ij} + \dot{s}^{ij}\mathcal{K}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + \dot{k}^i\gamma_i + k^i\dot{\gamma}_i \tag{13.8}$$

Next, we substitute (13.8) in the expression for power (12.3)

$$\rho_o\dot{\bar{\phi}} + \dot{\bar{\tau}}^{ij}\gamma_{ij} + \dot{s}^{ij}\mathcal{K}_{ij} + \dot{k}^i\gamma_i + \bar{\tau}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i = \bar{\tau}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i$$

or

$$\rho_o\left(\frac{\partial\bar{\phi}}{\partial\tau^{ij}}\dot{\bar{\tau}}^{ij} + \frac{\partial\bar{\phi}}{\partial s^{ij}}\dot{s}^{ij} + \frac{\partial\bar{\phi}}{\partial k^i}\dot{k}^i\right) + \gamma_{ij}\dot{\bar{\tau}}^{ij} + \mathcal{K}_{ij}\dot{s}^{ij} + \gamma_i\dot{k}^i = 0$$

or

$$\left(\gamma_{ij} + \rho_o\frac{\partial\phi}{\partial\tau^{ij}}\right)\dot{\bar{\tau}}^{ij} + \left(\mathcal{K}_{ij} + \rho_o\frac{\partial\bar{\phi}}{\partial s^{ij}}\right)\dot{s}^{ij} + \left(\gamma_i + \rho_o\frac{\partial\bar{\phi}}{\partial k^i}\right)\dot{k}^i = 0 \tag{13.9}$$

where we have assumed the rates $\dot{\bar{\tau}}^{ij}$, $\dot{s}^{ij}$ and $\dot{k}^i$ are all independent and their coefficients are rate independent. From (13.9) it follows

$$\gamma_{ij} = -\rho_o\frac{\partial\bar{\phi}}{\partial\tau^{ij}}$$

$$\mathcal{K}_{ij} = -\rho_o\frac{\partial\bar{\phi}}{\partial s^{ij}} \tag{13.10}$$

$$\gamma_i = -\rho_o\frac{\partial\bar{\phi}}{\partial k^i}$$

We note that the relationship between the Gibbs energy function ϕ, per unit mass of the compo-

site, and those of the constituents is given by

$$\rho_o G^{1/2}\phi = \frac{1}{\xi_2}\int_o^{\xi_2} \rho_o^* G^{*1/2}\phi^* d\xi \tag{13.11}$$

or

$$\rho_o\phi = \frac{1}{h_2}\int_o^{\xi_2} \nu(\rho_o^*\phi^*)d\xi = \frac{1}{h_2}\int_o^{h_2} \mu(\rho_o^*\phi^*)d\xi = \frac{1}{h_2}\int_o^{h_1} \mu(\rho_{o1}^*\phi_1^*)d\xi + \frac{1}{h_2}\int_{h_1}^{h_2} \mu(\rho_{o2}^*\phi_2^*)d\xi \tag{13.12}$$

where ϕ^* for an isotropic elastic material is given by (13.3). The explicit determination of the various coefficients in constitutive relations is beyond the scope of this project and is left for a follow-on project.

14. Some results for the case of a normal director

We recall that in the context of the present Cosserat composite theory, director **d** is a three dimensional vector associated with each composite particle and in general the only restriction placed on **d** is that it cannot be tangent to any ply. The case in which the director **D**, at each composite particle in the reference configuration, is taken to be the unit normal to the ply is of special interest. In such case, in order to allude to the direction of **D** in the present configuration, we may refer to director as "normal director." This section contains some results for the case of a normal director. The results of this section will be nelpful when we apply the theory to the cases of initially flat and initially cylindrical composite laminates. Therefore, in this section as well as in the rest of this development and within the context of the linearized theory we confine our attention to the case for which D is unit vector. Hence, we make the following kinematical assumptions in the reference (initial) configuration:

$$\mathbf{R}(\theta^\alpha,\theta^3) = \bar{\mathbf{R}}(\theta^\alpha) + \theta^3\mathbf{A}_3 \tag{14.1}$$

$$\mathbf{D} = \mathbf{A}_3 \tag{14.2}$$

and

$$\mathbf{P}^*(\theta^\alpha,\theta^3,\xi) = \mathbf{R}(\theta^\alpha,\theta^3) + \xi\mathbf{D}(\theta^\alpha,\theta^3) = \bar{\mathbf{R}} + (\theta^3 + \xi)\mathbf{A}_3 \tag{14.3}$$

where $\mathbf{P}^*$ is the position vector of an arbitrary point P^* of the micro-body, **R** is the position vector of the point P, corresponding to P^*, in the macro-body, and **D** is the director at point P. It is worth observing that in (14.1) and (14.3) the term involving ξ accounts for the effect of micro-structure while the term involving θ^3 represents the continuum behavior of the macro-structure, namely the composite laminate. In this connection it is important to realize that if, at the outset, in (14.3), we discard ξ with respect to θ^3 we will lose the effect of the micro-structure in the continuum formulation of composite laminates.

From (14.2) it follows

$$D_\alpha = 0 \quad , \quad D_3 = D(\eta^\alpha) = 1 \tag{14.4}$$

and

$$\zeta = \xi \tag{14.5}$$

From (14.1) we obtain

$$\begin{aligned} G_\alpha &= \frac{\partial \mathbf{R}}{\partial \theta^\alpha} = \mathbf{R}_{,\alpha} = \overline{\mathbf{R}}_{,\alpha} + \theta^3 \mathbf{A}_{3,\alpha} = \overline{\mathbf{R}}_{,\alpha} - \theta^3 B^\gamma{}_\alpha \mathbf{A}_\gamma = \overline{\mathbf{R}}_{,\alpha} - \theta^3 B^\gamma{}_\alpha \mathbf{G}_\gamma \\ G_3 &= \frac{\partial \mathbf{R}}{\partial \theta^3} = \mathbf{R}_{,3} = \mathbf{A}_3 \end{aligned} \tag{14.6}$$

Making use of (14.6) we write

$$\begin{aligned} G_{\alpha\beta} &= \overline{\mathbf{R}}_{,\alpha} \cdot \overline{\mathbf{R}}_{,\beta} - \theta^3 (B^\gamma{}_\alpha \overline{\mathbf{R}}_{,\beta} + B^\gamma{}_\beta \overline{\mathbf{R}}_{,\alpha}) \cdot \mathbf{A}_\gamma + (\theta^3)^2 B^\gamma{}_\alpha B^\delta{}_\beta A_{\gamma\delta} \\ G_{\alpha 3} &= (\overline{\mathbf{R}}_{,\alpha} - \theta^3 B^\gamma{}_\alpha \mathbf{A}_\gamma) \cdot \mathbf{A}_3 = 0 \\ G_{33} &= \mathbf{A}_3 \cdot \mathbf{A}_3 = 1 \end{aligned} \tag{14.7}$$

and

$$\begin{aligned} G^{1/2} &= [\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3] = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3 = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{A}_3 \\ &= (\overline{\mathbf{R}}_{,1} \times \overline{\mathbf{R}}_{,2}) \cdot \mathbf{A}_3 - \theta^3 \beta^\gamma{}_1 (\mathbf{A}_\gamma \times \overline{\mathbf{R}}_{,2}) \cdot \mathbf{A}_3 - \theta^3 B^\gamma{}_2 (\overline{\mathbf{R}}_{,1} \times \mathbf{A}_\gamma) \cdot \mathbf{A}_3 \\ &\quad + (\theta^3)^2 B^\gamma{}_1 B^\delta{}_2 (\mathbf{A}_\gamma \times \mathbf{A}_\delta) \cdot \mathbf{A}_3 \end{aligned} \tag{14.8}$$

Moreover, from (14.2) and (14.3) it follows

$$\begin{aligned} G^*_\alpha &= \frac{\partial \mathbf{P}^*}{\partial \eta^\alpha} = \frac{\partial \mathbf{P}^*}{\partial \theta^\alpha} = \mathbf{R}_{,\alpha} + \xi \mathbf{A}_{3,\alpha} = \mathbf{R}_{,\alpha} - \xi B^\gamma{}_\alpha \mathbf{A}_\gamma = \nu^\gamma{}_\alpha \mathbf{G}_\gamma \\ G^*_3 &= \frac{\partial \mathbf{P}^*}{\partial \xi} = \mathbf{A}_3 \end{aligned} \tag{14.9}$$

where

$$\nu^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \xi B^\gamma{}_\alpha = \mu^\gamma{}_\alpha \tag{14.10}$$

and

$$\nu = D\,\det(\nu^{\gamma}{}_{\alpha}) = \det(\nu^{\gamma}{}_{\alpha}) = \det(\mu^{\gamma}{}_{\alpha}) = \mu \tag{14.11}$$

Making use of (14.9), we obtain

$$G^{*}_{\alpha\beta} = \nu^{\gamma}{}_{\alpha}\nu^{\delta}{}_{\beta}G_{\gamma\delta} = \mu^{\gamma}{}_{\alpha}\mu^{\delta}{}_{\beta}G_{\gamma\delta}$$

$$G^{*}_{\alpha3} = 0 \tag{14.12}$$

$$G^{*}_{33} = 1$$

and

$$G^{*1/2} = G^{1/2}\det(\nu^{\alpha}{}_{\beta}) = \nu G^{1/2} = \mu G^{1/2}$$

or

$$\nu = \left(\frac{G^{*}}{G}\right)^{1/2} = \mu \tag{14.13}$$

In view of (14.1) to (14.3) and (14.10) to (14.13) expressions (11.40) are reduced to

$$\tau^{\alpha\beta} = \frac{1}{h_2}\int_{0}^{\xi_2=h_2} \nu\tau^{*\alpha\gamma}\nu^{\beta}{}_{\gamma}d\xi \quad , \quad \tau^{\alpha3} = \frac{1}{h_2}\int_{0}^{\xi_2=h_2} \nu\tau^{*\alpha3}d\xi$$

τ^{3i} or $\mathbf{T}^3$ are specified by a constitutive equation directly.

$$s^{\alpha\beta} = \frac{1}{h_2}\int_{0}^{\xi_2=h_2} \nu\tau^{*\alpha\gamma}\nu^{\beta}{}_{\gamma}\xi d\xi \quad , \quad s^{\alpha3} = \frac{1}{h_2}\int_{0}^{\xi_2=h_2} \nu\tau^{*\alpha3}\xi d\xi$$

$$s^{31} = 0 \qquad \text{or} \qquad \mathbf{S}^3 = 0 \tag{14.14}$$

$$k^{\alpha} = \frac{1}{h_2}\int_{0}^{\xi_2=h_2} \nu\tau^{*3\gamma}\nu^{\alpha}{}_{\gamma}d\xi \quad , \quad k^{3} = \frac{1}{h_2}\int_{0}^{\xi_2=h_2} \nu\tau^{*33}d\xi$$

while definitions (11.49) become

$$\bar{\tau}^{\alpha\beta} = \frac{1}{h_2}\int_0^{h_2} \mu\bar{\tau}^{*\alpha\gamma}\mu^{\beta}{}_{\gamma}d\xi \quad , \quad v^{\alpha} = \bar{\tau}^{\alpha 3} = \frac{1}{h_2}\int_0^{h_2} \mu\bar{\tau}^{*\alpha 3}d\xi$$

$$\bar{\tau}^{3\alpha} = \tau^{3\alpha} \quad , \quad \bar{\tau}^{33} = \tau^{33}$$

$$\bar{s}^{\alpha\beta} = \frac{1}{h_2}\int_0^{h_2} \mu\bar{\tau}^{*\alpha\gamma}\mu^{\beta}{}_{\gamma}\xi d\xi \quad , \quad \bar{s}^{\alpha 3} = \frac{1}{h_2}\int_0^{h_2} \mu\bar{\tau}^{*\alpha 3}\xi d\xi \tag{14.15}$$

$$v^3 = \frac{1}{h_2}\int_0^{h_2} \mu(\bar{\tau}^{*33} - B_{\alpha\beta}\bar{\tau}^{*\alpha\gamma}\mu^{\beta}{}_{\gamma}\xi)d\xi$$

Also, the transformation between $\bar{\tau}^{*ij}$ and τ^{*ij}, namely expressions (14.14) to (14.16), are now given by

$$\bar{\tau}^{*ij} = \tau^{*ij} \tag{14.16}$$

Finally the relations between the two sets of definitions (14.14) and (14.15) are reduced to

$$\bar{\tau}^{\alpha\beta} = \tau^{\alpha\beta} \qquad v^{\alpha} = \bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = k^{\alpha} + B^{\alpha}{}_{\gamma}s^{\gamma 3}$$
$$\bar{s}^{ij} = s^{ij} \qquad v^3 = k^3 - B_{\alpha\beta}s^{\alpha\beta} \tag{14.17}$$

15. Theory of initially flat composite laminates

We are now in a position to apply the theory of Cosserat composite to special initial geometric configurations. In this section we apply the theory to the case of an initially flat composite laminate. The case of an initially cylindrical composite laminate will be considered in the next section.

Consider a composite laminate and let its plies be flat (i.e., having no curvature) in the reference (initial) configuration. Let $\mathbf{e}_i$ (i = 1,2,3) be the base vectors associated with a system of Cartesian coordinates x_i (i= 1,2,3). The position vector of a plane surface perpendicular to $\mathbf{e}_3$ and passing through the point (0,0,c) may be specified by

$$\mathbf{p}(x^i) = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + c\mathbf{e}_3 \tag{15.1}$$

where c is a constant. In view of (15.1) and recalling formulae (14.1) to (14.3) we adopt the following kinematical assumptions for an initially flat composite laminate:

$$\mathbf{R}(x^\alpha,x^3) = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \tag{15.2}$$

$$\mathbf{D} = \mathbf{A}_3 = \mathbf{e}_3 \tag{15.3}$$

and

$$\mathbf{P}^*(x^\alpha,x^3,\xi) = \mathbf{R} + \zeta\mathbf{e}_e = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + (x^3 + \zeta)\mathbf{e}_3 \tag{15.4}$$

We recall that (15.2) specifies the position of an arbitrary macro-particle of the composite laminate while the position vector of the micro-particle corresponding to the macro-particle is given by (15.4).

First we proceed to obtain various quantities associated with the surface (15.2). The base vectors of the surface are obtained from (15.2) as follows:

$$A_\alpha = \frac{\partial R}{\partial x^\alpha}$$
$$A_1 = \frac{\partial R}{\partial x^1} = e_1 \quad , \quad A_2 = \frac{\partial R}{\partial x^2} = e_2 \tag{15.5}$$

The components of the surface metric tensor are

$$A_{\alpha\beta} = A_\alpha \cdot A_\beta$$
$$A_{11} = e_1 \cdot e_1 = 1 \quad , \quad A_{12} = A_{21} = A_1 \cdot A_2 = 0 \quad , \quad A_{22} = e_2 \cdot e_2 = 1 \tag{15.6.a}$$

or

$$(A_{\alpha\beta}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{15.6.b}$$

Moreover, we have

$$A^{\alpha\beta}A_{\beta\gamma} = \delta^\alpha{}_\gamma \Rightarrow A^{\alpha\beta} = (A_{\alpha\beta})^{-1} \tag{15.7}$$

Hence,

$$A^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{15.8}$$

The conjugate base vectors of the surface are given by

$$A^\alpha = A^{\alpha\beta}A_\beta$$

Therefore

$$A^1 = A^{11}A_1 + A^{12}A_2 = A_1 = e_1 \quad , \quad A^2 = A^{21}A_1 + A^{22}A_2 = A_2 = e_2 \tag{15.9}$$

The unit normal to the surface follows from (15.9):

$$A_3 = \frac{A_1 \times A_2}{|A_1 \times A_2|} = e_3 \tag{15.10}$$

which confirms (15.3). In view of (15.5.b) and the expressions for Christoffel symbols of the first and second kind, i.e.,

$$[\alpha\beta,\gamma] = \frac{1}{2}\left(\frac{\partial A_{\gamma\beta}}{\partial x^{\alpha}} + \frac{\partial A_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial A_{\alpha\beta}}{\partial x^{\gamma}}\right)$$

and (15.11)

$$\{{}_{\alpha}{}^{\gamma}{}_{\beta}\} = a^{\gamma\delta}[\alpha\beta,\delta]$$

In view of (15.6.b), it is clear that all Christoffel symbols vanish, i.e.,

$$[\alpha\beta,\gamma] = \{{}_{\alpha}{}^{\gamma}{}_{\beta}\} = 0 \tag{15.12}$$

Coefficients of the second fundamental form of the surface are given by

$$\mathbf{B}_{\alpha\beta} = \mathbf{A}_{\alpha,\beta} \cdot \mathbf{A}_3 = -\,\mathbf{A}_{\beta} \cdot \mathbf{A}_{3,\alpha} \tag{15.13}$$

It then follows from (15.4) and (15.13) that

$$B_{\alpha\beta} = B^{\alpha}{}_{\beta} = 0 \tag{15.14}$$

This shows that for an initially flat ply (plate) the components of the second fundamental form of the surface vanish identically.

Next, we obtain the various kinematical quantities associated with micro and macro continuua for the case of initially flat composite laminate. From (14.6) it follows

$$\mathbf{G}_{\alpha} = \frac{\partial \mathbf{R}}{\partial \theta^{\alpha}} = \frac{\partial \mathbf{R}}{\partial x^{\alpha}} = \mathbf{e}_{\alpha}$$

(15.15)

$$\mathbf{G}_3 = \frac{\partial \mathbf{R}}{\partial \theta^3} = \frac{\partial \mathbf{R}}{\partial x^3} = \mathbf{e}_3$$

From (15.15) we obtain

$$G_{\alpha\beta} = \mathbf{R}_{,\alpha} \cdot \mathrm{RT}_{,\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$$

$$G_{\alpha 3} = \mathbf{R}_{,\alpha} \cdot \mathbf{G}_3 = \mathbf{e}_\alpha \cdot \mathbf{e}_3 = 0 \qquad (15.16.a)$$

$$G_{33} = \mathbf{G}_3 \cdot \mathbf{G}_3 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 0$$

or

$$(G_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (15.16.b)$$

Also,

$$G^{1/2} = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3 = (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = 1 \qquad (15.17)$$

Moreover, from (14.9), (15.4) and (15.14) we have

$$\mathbf{G}^*_\alpha = \frac{\partial \mathbf{P}^*}{\partial \eta^\alpha} = \frac{\partial \mathbf{P}^*}{\partial \theta^\alpha} = \mathbf{R}_{,\alpha} - \xi B^\gamma{}_\alpha \mathbf{A}_\gamma = \mathbf{e}_\alpha$$

$$\mathbf{G}^*_3 = \frac{\partial \mathbf{P}^*}{\partial \zeta} = \frac{\partial \mathbf{P}^*}{\partial \xi} = \mathbf{A}_3 = \mathbf{e}_3 \qquad (15.18)$$

Also, from (14.10) and (14.11) we obtain

$$\nu^\gamma{}_\alpha = \mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha \qquad (15.19)$$

and

$$\nu = D \det(\nu^\gamma{}_\alpha) = \mu = 1 \qquad (15.20)$$

Making use of (15.18), we obtain

$$G^*_{\alpha\beta} = \mathbf{G}^*_\alpha \cdot \mathbf{G}^*_\beta = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$$

$$G^*_{\alpha 3} = \mathbf{G}^*_\alpha \cdot \mathbf{G}^*_3 = \mathbf{e}_\alpha \cdot \mathbf{e}_3 = 0 \qquad (15.21.a)$$

$$G^*_{33} = \mathbf{G}^*_3 \cdot \mathbf{G}^*_3 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

or

$$(G^*_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15.21.b}$$

It then follows that

$$G^{*1/2} = (G^*_1 \times G^*_2) \cdot G^*_3 = (e_1 \times e_2) \cdot e_3 = 1 \tag{15.22}$$

By (15.17) and (15.22) we have

$$(\frac{G^*}{G})^{1/2} = 1 = \nu = \mu \tag{15.23}$$

which confirms (15.20). In view of (15.2) to (15.4) and (15.23) formulae (14.14) and (14.15) simplify as follows:

$$\bar{\tau}^{\alpha\beta} = \tau^{\alpha\beta} = \frac{1}{h_2}\int_o^{h_2} \tau^{*\alpha\beta} d\zeta \quad , \quad \bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = v^\alpha = k^\alpha = \frac{1}{h_2}\int_o^{h_2} \tau^{*\alpha 3} d\zeta$$

$\bar{\tau}^{3i} = \tau^{3i}$ be specified by a constitutive relation directly

$$\bar{s}^{\alpha\beta} = s^{\alpha\beta} = \frac{1}{h_2}\int_o^{h_2} \tau^{*\alpha\beta}\zeta d\zeta \quad , \quad \bar{s}^{\alpha 3} = s^{\alpha 3} = \frac{1}{h_2}\int_o^{h_2} \tau^{*\alpha 3}\zeta d\zeta \tag{15.24}$$

$$s^{3i} = 0$$

$$k^3 = v^3 = \frac{1}{h_2}\int_o^{h_2} \tau^{*33} d\zeta$$

where in obtaining formulae (15.24) we have noticed that

$$\bar{\tau}^{*ij} = \tau^{*ij} \tag{15.25}$$

where τ^{*ij} are now Cartesian components of the classical stress tensor.

We recall at this point that because all quantities are now referred to rectangular Cartesian axes, covariant differentiation with respect to metric tensory G_{ij} is reduced to partial differentiation with respect to x^i (or x_i)and no distinction needs to be made between superscripts and sub-

scripts. In view of this, expressions (9.30) to (9.32) are reduced to [1]

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{15.26}$$

$$\gamma_i = \delta_i + u_{3,i} \tag{15.27}$$

$$\mathcal{K}_{ij} = \delta_{i,j} \tag{15.28}$$

Finally, with the help of (15.3) equations of motion for the case of an initially composite laminante are reduced to [1]

$$\tau_{ij,i} + \rho_o b_j = \rho_o(\ddot{u}_j + y^1\ddot{\delta}_j) \tag{15.29}$$

$$s_{ij,i} + (\rho_o c_j - k_j) = \rho_o(y^1\ddot{\delta}_j + y^2\ddot{\delta}_j) \tag{15.30}$$

$$\varepsilon_{ijn}\tau_{ij} = 0 \tag{15.31}$$

We observe that in (15.31), ε_{ijn} is skew-symmetric with respect to i and j; hence it follows that

$$\tau_{ij} = \tau_{ji} \tag{15.32}$$

This indicates that in the case of an initially flat composite laminate the components of the composite stress tensor are symmetric. The same conclusion can be reached from expressions $(15.24)_{1,2}$, i.e.,

$$\tau^{ij} = \tau_{ij} = \frac{1}{h_2}\int_0^{h_2} \tau^*_{ij}\,d\xi$$

in view of the symmetry of the classical stress tensor.

[1] In expressions like (15.27) the Greek letters such as δ_i denote components of the director displacement vectors, etc. This should not be confused with the use of Greek letters as indices in various expressions.

16. Theory of initially cylindrical composite laminates

In this section we continue to apply the theory of Cosserat composite to initially cylindrical composite laminates.

Consider a composite laminate and let its plies form a set of concentric right circular cylindrical surfaces. Let x^i (i = 1,2,3) and $\{r,\theta,z\}$ denote Cartesian and cylindrical coordinates with a common origin in a Euclidean three-dimensional space. Let $\mathbf{e}_i$ (i = 1,2,3) and $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$ denote the unit base vectors in the foregoing coordinate systems, respectively. We recall that a right circular cylinder of radius r may be defined by a position vector of the form

$$\mathbf{P} = r\mathbf{e}_r + z\mathbf{e}_z \tag{16.1}$$

Recalling the relations between the unit base vectors in Cartesian and cylindrical coordinate systems, i.e.,

$$\begin{aligned} \mathbf{e}_r &= \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2 \\ \mathbf{e}_\theta &= -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2 \\ \mathbf{e}_z &= \mathbf{e}_3 \end{aligned} \tag{16.2}$$

we obtain

$$\mathbf{P} = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + z\mathbf{e}_3 \tag{16.3}$$

It is worth mentioning that sometimes it is more convenient to consider an alternative representation of the cylindriccal surface (16.3) as follows:

$$\mathbf{P} = (r\cos\tfrac{s}{r})\mathbf{e}_1 + (r\sin\tfrac{s}{r})\mathbf{e}_2 + z\mathbf{e}_3 \tag{16.4}$$

where $s = r\theta$ is the arclength measured from a fixed point ($\theta = 0$) along the section curve. Let us now introduce a set of coordinates θ^i (i = 1,2,3) such that

$$\theta^1 = r\theta \quad , \quad \theta^2 = z \quad , \quad \theta^3 = r \tag{16.5}$$

Hence, in terms of θ^i coordinates we have

$$\mathbf{P} = (\theta^3 \cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\theta^3 \sin \frac{\theta^1}{\theta^3})\mathbf{e}_2 + \theta^2\mathbf{e}_3 \tag{16.6}$$

This representation will facilitate much of the intermediate steps especially in connection to calculation of the various quantities of the surface.

In view of the foregoing explanation, we now adopt the following kinematical assumptions for an initially cylindrical composite laminate

$$\mathbf{R}(r,\theta,z) = r\mathbf{e}_r + z\mathbf{e}_z$$

$$\mathbf{D} = \mathbf{A}_3 = \mathbf{e}_r \tag{16.7}$$

$$\mathbf{P}^*(r,\theta,z,\zeta) = (r + \zeta)\mathbf{e}_r + z\mathbf{e}_z$$

Making use of (16.5), we can rewrite this

$$\mathbf{R}(\theta^\alpha,\theta^3) = (\theta^3 \cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\theta^3 \sin \frac{\theta^1}{\theta^3})\mathbf{e}_2 + \theta^2\mathbf{e}_3$$

$$\mathbf{D} = \mathbf{A}_3 = (\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2 \tag{16.8}$$

$$\mathbf{P}^*(\theta^\alpha,\theta^3,\zeta) = [(\theta^3 + \zeta)\cos \frac{\theta^1}{\theta^3}]\mathbf{e}_1 + [(\theta^3 + \zeta)\sin \frac{\theta^1}{\theta^3}]\mathbf{e}_2 + \theta^2\mathbf{e}_3$$

The base vectors of the surface are obtained from $(16.8)_1$ as follows

$$\mathbf{A}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha}$$

Hence we have

$$\mathbf{A}_1 = \mathbf{R}_{,\theta^1} = \frac{1}{\theta^3}\,(-\,\theta^3 \sin \frac{\theta^1}{\theta^3})\mathbf{e}_1 + \frac{1}{\theta^3}\,(\theta^3 \cos \frac{\theta^1}{\theta^3})\mathbf{e}_2$$

$$= -\,(\sin \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos \frac{\theta^1}{\theta^3})\mathbf{e}_2 = \mathbf{e}_\theta \qquad (16.9)$$

and

$$\mathbf{A}_2 = \mathbf{R}_{,\theta^2} = \mathbf{e}_3 = \mathbf{e}_z \qquad (16.10)$$

From (16.9) and (16.10) we obtain the components of the surface metric tensor $A_{\alpha\beta}$

$$A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta$$

Therefore

$$A_{11} = \mathbf{A}_1 \cdot \mathbf{A}_1 = [-(\sin \theta^1)\mathbf{e}_1 + \cos(\theta^1)\mathbf{e}_2] \cdot [-(\sin \theta^1)\mathbf{e}_1 + (\cos \theta^1)\mathbf{e}_2] = 1$$

$$A_{12} = A_{21} = \mathbf{A}_1 \cdot \mathbf{A}_2 = [-(\sin \theta^1)\mathbf{e}_1 + (\cos \theta^1)\mathbf{e}_2] \cdot \mathbf{e}_3 = 0 \qquad (16.11.a)$$

$$A_{22} = \mathbf{A}_2 \cdot \mathbf{A}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

or

$$(A_{\alpha\beta}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (16.11.b)$$

Moreover, we have

$$A^{\alpha\beta}A_{\beta\gamma} = \delta^\alpha{}_\gamma \;\Rightarrow\; A^{\alpha\beta} = (A_{\alpha\beta})^{-1} \qquad (16.12)$$

Hence,

$$A^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (16.13)$$

The conjugate base vectors of the surface are given by

$$\mathbf{A}^\alpha = A^{\alpha\beta}\mathbf{A}_\beta$$

Thus,

$$A^1 = A^{11}A_1 + A^{12}A_2 = A_1 = e_\theta$$

$$A^2 = A^{21}A_1 + A^{22}A_2 = A_2 = e_3 = e_z \tag{16.14}$$

The unit normal to the surface follows from (16.14)

$$A_3 = \frac{A_1 \times A_2}{|A_1 \times A_2|} = \frac{1}{|A_1 \times A_2|}\{[-(\sin\theta^1)e_1 + (\cos\theta^1)e_2] \times e_3\}$$

or

$$A_3 = \frac{1}{|A_1 \times A_2|}\{-(\sin\frac{\theta^1}{\theta^3})(e_1 \times e_3) + (\cos\frac{\theta^1}{\theta^3})(e_2 \times e_3)\}$$

$$= \frac{1}{|A_1 \times A_2|}\{-(\cos\frac{\theta^1}{\theta^3})e_1 + (\sin\frac{\theta^1}{\theta^3})e_2\}$$

$$= (\cos\frac{\theta^1}{\theta^3})e_1 + (\sin\frac{\theta^1}{\theta^3})e_2 = e_r$$

We note that A_3 could have been obtained from vector product of e_θ and e_z. However, to illustrate the general procedure we did not make use of e_θ and e_z. The Christoffel symbols of the first and second kind follow from (16.11)

$$[\alpha\beta,\gamma] = \{{}_\alpha{}^\gamma{}_\beta\} = 0 \tag{16.16}$$

and coefficients of the second fundamental form of the surface are given by

$$B_{\alpha\beta} = A_{\alpha,\beta} \cdot A_3 = -A_\alpha \cdot A_{3,\alpha}$$

hence,

$$B_{11} = \mathbf{A}_{1,1} \cdot \mathbf{A}_3 = [-(\sin \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos \frac{\theta^1}{\theta^3})\mathbf{e}_2]_{,1} \cdot [(\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2]$$

$$= \frac{1}{\theta^3} [-(\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 - (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot [(\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2] = -\frac{1}{\theta^3} = -\frac{1}{r}$$

$$B_{12} = \mathbf{A}_{2,1} \cdot \mathbf{A}_3 = (\mathbf{e}_3)_{,1} \cdot [(\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

$$B_{21} = \mathbf{A}_{1,2} \cdot \mathbf{A}_3 = [-(\sin \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos \frac{\theta^1}{\theta^3}\mathbf{e}_{]_{,2}} \cdot [\cos (\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

$$B_{22} = \mathbf{A}_{2,2} \cdot \mathbf{A}_3 = (\mathbf{e}_3)_{,2} \cdot [(\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

Therefore

$$(B_{\alpha\beta}) = \begin{bmatrix} -1/\theta^3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/r & 0 \\ 0 & 0 \end{bmatrix} \tag{16.17}$$

We also have

$$B^{\alpha}{}_{\beta} = A^{\alpha\gamma}B_{\gamma\beta}$$

Hence,

$$B^1{}_1 = A^{11}B_{11} + A^{12}B_{21} = -\frac{1}{\theta^3} = -\frac{1}{r}$$

$$B^1{}_2 = A^{11}B_{21} + A^{12}B_{22} = 0$$

$$B^2{}_1 = A^{21}B_{11} + A^{22}B_{21} = 0$$

$$B^2{}_2 = A^{21}B_{12} + A^{22}B_{22} = 0$$

or

$$(B^{\alpha}{}_{\beta}) = \begin{bmatrix} -1/\theta^3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/r & 0 \\ 0 & 0 \end{bmatrix} \tag{16.18}$$

Next, we obtain the various kinematical quantities associated with micro and macro continua for

the case of initially cylindrical composite laminates. From (14.6) and (16.8) it follows that

$$\mathbf{G}_1 = \frac{\partial \mathbf{R}}{\partial \theta^1} = -(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2 = \mathbf{e}_\theta$$

$$\mathbf{G}_2 = \frac{\partial \mathbf{R}}{\partial \theta^2} = \mathbf{e}_3 = \mathbf{e}_z \tag{16.19}$$

$$\mathbf{G}_3 = \frac{\partial \mathbf{R}}{\partial \theta^3} = (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2 = \mathbf{e}_r$$

From (16.19) we obtain

$$G_{11} = \mathbf{G}_1 \cdot \mathbf{G}_1 = [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 1$$

$$G_{12} = G_{21} = \mathbf{G}_1 \cdot \mathbf{G}_2 = [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot \mathbf{e}_3 = 0$$

$$G_{13} = G_{31} = \mathbf{G}_1 \cdot \mathbf{G}_3 = [-(\sin\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

$$G_{22} = \mathbf{G}_2 \cdot \mathbf{G}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

$$G_{23} = G_{32} = \mathbf{G}_2 \cdot \mathbf{G}_3 = \mathbf{e}_3 \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 0$$

$$G_{33} = \mathbf{G}_3 \cdot \mathbf{G}_3 = [(\cos\frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] \cdot [(\cos\frac{\theta^1}{\theta^3})\mathbf{e} + (\sin\frac{\theta^1}{\theta^3})\mathbf{e}_2] = 1$$

Hence

$$(G_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16.20}$$

We also have

$$G^{1/2} = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3$$

$$= \{[-(\sin \frac{\theta^1}{\theta^3})\, \mathbf{e}_1 + (\cos \frac{\theta^1}{\theta^3})\, \mathbf{e}_2] \times \mathbf{e}_3\} \cdot [(\cos \frac{\theta^1}{\theta^3})\, \mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\, \mathbf{e}_2]$$

$$= \{[-(\sin \frac{\theta^1}{\theta^3})(-\mathbf{e}_2) + (\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1] \cdot [(\cos \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\sin \frac{\theta^1}{\theta^3})\mathbf{e}_2] = 1 \qquad (16.21)$$

Moreover, from (14.9) and (16.7) we have

$$\mathbf{G}_1^* = \frac{\partial \mathbf{P}^*}{\partial \theta^1} = \frac{\theta^3 + \zeta}{\theta^3} [-(\sin \frac{\theta^1}{\theta^3})\mathbf{e}_1 + (\cos \frac{\theta^1}{\theta^3})\mathbf{e}_2] = \frac{r+\zeta}{r}\, \mathbf{e}_\theta$$

$$\mathbf{G}_2^* = \frac{\partial \mathbf{P}^*}{\partial \theta^2} = \mathbf{e}_3 = \mathbf{e}_z \qquad (16.22)$$

$$\mathbf{G}_3^* = \frac{\partial \mathbf{P}^*}{\partial \xi} = \frac{\partial \mathbf{P}^*}{\partial \zeta} = [(\cos \theta^1)\mathbf{e}_1 + (\sin\theta^1)\mathbf{e}_2] = \mathbf{e}_r$$

Also, from (14.10), (14.11) and (16.18) we obtain

$$\nu^\gamma{}_\alpha = \mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \xi B^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \zeta B^\gamma{}_\alpha$$

$$\nu^1{}_1 = 1 - \zeta(-\frac{1}{\theta^3}) = 1 + \frac{\zeta}{r} = \frac{r+\zeta}{r}$$

$$\nu^1{}_2 = \nu^2{}_1 = 0$$

$$\nu^2{}_2 = 1$$

or

$$(\nu^\gamma{}_\alpha) = (\mu^\gamma{}_\alpha) = \begin{bmatrix} (r+\zeta)/r & 0 \\ 0 & 1 \end{bmatrix} \qquad (16.23)$$

and

$$\nu = D \det(\nu^\gamma{}_\alpha) = \det(\nu^\gamma{}_\alpha) = \mu = \frac{r+\zeta}{r} \qquad (16.24)$$

Making use of (16.22) we obtain

$$G^*_{11} = \mathbf{G}^*_1 \cdot \mathbf{G}^*_1 = (\frac{\theta^3+\zeta}{\theta^3})^2[-(\sin\theta^1)\mathbf{e}_1 + (\cos\theta^1)\mathbf{e}_2] \cdot [-(\sin\theta^1)\mathbf{e}_1 + (\cos\theta^1)\mathbf{e}_2] = (\frac{\theta^3+\zeta}{\theta^3})^2 = (\frac{r+\zeta}{r})^2$$

$$G^*_{12} = G^*_{21} = \mathbf{G}^*_1 \cdot \mathbf{G}^*_2 = (\frac{\theta^3+\zeta}{\theta^3})[-(\sin\theta^1)\mathbf{e}_1 + (\cos\theta^1)\mathbf{e}_2] \cdot \mathbf{e}_3 = 0$$

$$G^*_{13} = G^*_{31} = \mathbf{G}^*_1 \cdot \mathbf{G}^*_3 = (\frac{\theta^3+\zeta}{\theta^3})[-(\sin\theta^1)\mathbf{e}_1 + (\cos\theta^1)\mathbf{e}_2] \cdot [-(\cos\theta^1)\mathbf{e}_1 + (\sin\theta^1)\mathbf{e}_2] = 0$$

$$G^*_{22} = \mathbf{G}^*_2 \cdot \mathbf{G}^*_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$$

$$G^*_{23} = G^*_{32} = \mathbf{G}^*_2 \cdot \mathbf{G}^*_3 = \mathbf{e}_3 \cdot [(\cos\theta^1)\mathbf{e}_1 + (\sin\theta^1)\mathbf{e}_2] = 0$$

$$G^*_{33} = \mathbf{G}^*_3 \cdot \mathbf{G}^*_3 = [(\cos\theta^1)\mathbf{e}_1 + (\sin\theta^1)\mathbf{e}_2] \cdot [(\cos\theta^1)\mathbf{e}_1 + (\sin\theta^1)\mathbf{e}_2] = 1$$

Hence,

$$(G^*_{ij}) = \begin{bmatrix} (1+\zeta/\theta^3)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1+\zeta/r)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16.25}$$

It then follows that

$$G^{*1/2} = \{\det(G^*_{ij})\}^{1/2} = (\frac{\theta^3+\zeta}{\theta^3}) = \frac{r+\zeta}{r} \tag{16.26}$$

By (16.21) and (16.26) we have

$$(\frac{G^*}{G})^{1/2} = \frac{r+\zeta}{r} = \nu = \mu \tag{16.27}$$

which confirms (16.24). In view of (16.8) and (16.27) formulae (14.14) and (14.15) reduce to

$$\bar{\tau}^{\alpha 1} = \tau^{\alpha 1} = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})\tau^{*\alpha\gamma}\nu^1{}_\gamma d\zeta = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha 1} d\zeta \tag{16.28.a}$$

$$\bar{\tau}^{\alpha 2} = \tau^{\alpha 2} = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})\tau^{*\alpha\gamma}\nu^2{}_\gamma d\zeta = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})\tau^{*\alpha 2} d\zeta \tag{16.28.b}$$

$$\bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})\tau^{*\alpha 3}d\zeta = v^{\alpha} \tag{16.28.c}$$

$\bar{\tau}^{3i}$ be specified by a constitutive relation directly (16.28.d)

$$\bar{s}^{\alpha 1} = s^{\alpha 1} = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})\tau^{*\alpha\gamma}v^1{}_{,\gamma}\zeta d\zeta = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})^2\tau^{*\alpha 1}\zeta d\zeta \tag{16.28.e}$$

$$\bar{s}^{\alpha 2} = s^{\alpha 2} = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})\tau^{*\alpha\gamma}v^2{}_{,\gamma}\zeta d\zeta = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})\tau^{*\alpha 2}\zeta d\zeta \tag{16.28.f}$$

$$\bar{s}^{\alpha 3} = s^{\alpha 3} = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})\tau^{*\alpha 3}\zeta d\zeta \tag{16.28.g}$$

$$s^{3i} = 0 \quad \text{or} \quad S^3 = 0 \tag{16.28.h}$$

$$k^1 = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})^2\tau^{*31}d\zeta \ , \ k^2 = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})\tau^{*32}d\zeta \tag{16.28.i}$$

$$k^3 = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})\tau^{*33}d\zeta \tag{16.28.j}$$

$$v^3 = \frac{1}{h_2}\int_o^{h_2}(1+\frac{\zeta}{r})[\tau^{*33} - (1+\frac{\zeta}{r})\tau^{*11}\frac{\zeta}{r}]d\zeta \tag{16.28.k}$$

It is interesting to observe that when the radius of the cylindrical laminate becomes large (i.e., when the cylindrical surface approaches a flat surface) the value of $\frac{\zeta}{r}$ becomes small and may be neglected in comparison to unity (ideally $\frac{\zeta}{r}$ approaches zero) and the various expression obtained in this section will reduce to those obtained for an initially flat composte laminate.

The relative kinematical measures γ_{ij}, γ_i and $\mathcal{K}_{ij}$ are now given by

$$\gamma_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i}) \tag{16.29}$$

$$\gamma_i = \delta_i + u_{3|i} \tag{16.30}$$

$$\mathcal{K}_{ij} = \delta_{i|j} \tag{16.31}$$

where a vertical bar (|) denotes covariant differentiation with respect to coordinates θ^i (i = 1,2,3) as specified by (16.5). Moreover, equations of motion are given by

$$\tau_j^i{}_{|i} + \rho_o b_j = \rho_o(\alpha_j + y^1\beta_j) \tag{16.32}$$

$$s_j^i{}_{|i} + (\rho_o c_j - k_j) = \rho_o(y^1\alpha_j + y^2\beta_j) \tag{16.33}$$

where all components in the above are referred to coordinates θ^i (i = 1,2,3).

For convenience and systematic reduction of various results of this section we adopted the coordinate system (16.5). However, most of the available results in continuum mechanics regarding cylindrical bodies are in terms of the cylindrical coordinates r,θ,z. In order to write the relevant results of this section in terms of r,θ,z we consider the representation (16.7) and adopt a system of cylindrical coordinates r,θ,z such that

$$\theta^1 = \theta \quad , \quad \theta^2 = z \quad , \quad \theta^3 = r \tag{16.34}$$

From $(16.7)_1$ and (16.34) it follows

$$\mathbf{G}_1 = r\mathbf{e}_\theta \quad , \quad \mathbf{G}_2 = \mathbf{e}_z \quad , \quad \mathbf{G}_3 = \mathbf{e}_r \tag{16.35}$$

and

$$(G_{ij}) = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (G^{ij}) = \begin{bmatrix} 1/r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16.36}$$

$$G^{1/2} = r \tag{16.37}$$

Moreover, from $(16.7)_3$ we obtain

$$\mathbf{G}_1^* = (r+\zeta)\mathbf{e}_\theta \quad , \quad \mathbf{G}_2 = \mathbf{e}_z \quad , \quad \mathbf{G}_3 = \mathbf{e}_r \tag{16.38}$$

and

$$(G^*_{ij}) = \begin{bmatrix} (r+\zeta)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (G^{*ij}) = \begin{bmatrix} 1/(r+\zeta)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16.39}$$

$$G^{*1/2} = (r+\zeta) \tag{16.40}$$

From (14.13), (16.37) and (16.40) it follows

$$\nu = \mu = (\frac{G^*}{G})^{1/2} = \frac{r+\zeta}{r} = (1 + \frac{\zeta}{r}) \tag{16.41}$$

as before. In order to calculate expresisons involving covariant differentiation we need to calculate the Christoffel symbols of the first and second kind. Christoffel symbols of the first kind are given by

$$[ijk] = \frac{1}{2}\,(g_{jk,i} + g_{ki,j} - g_{ij,k}) \tag{16.42}$$

The only non-vanishing Christoffel symbols of the first kind are

$$[311] = r \quad , \quad [131] = r \quad , \quad [113] = -\,r \tag{16.43}$$

Christoffel symbols of the second kind are given by

$$\{_i{}^k{}_j\} = g^{km}[ijm] \tag{16.44}$$

From (16.43) and (16.44) the only non-vanishing Christoffel symbols of the second kind are

$$\{_1{}^3{}_1\} = -\,r \quad , \quad \{_1{}^1{}_3\} = \frac{1}{r} \quad , \quad \{_3{}^1{}_1\} = \frac{1}{r} \tag{16.45}$$

The physical components of the displacement vector **u** and the director displacement δ are given by

$$\bar{u}_1 = (g_{11})^{-1/2}u_1 = (g_{11})^{1/2}u^1 = u_\theta$$

$$\bar{u}_2 = (g_{22})^{-1/2}u_2 = (g_{22})^{1/2}u^2 = u_z \tag{16.46}$$

$$\bar{u}_3 = (g_{33})^{-1/2}u_3 = (g_{33})^{1/2}u^3 = u_r$$

$$\bar{\delta}_1 = (g_{11})^{-1/2}\delta_1 = (g_{11})^{1/2}\delta^1 = \delta_\theta$$

$$\bar{\delta}_2 = (g_{22})^{-1/2}\delta_2 = (g_{22})^{1/2}\delta^2 = \delta_z \tag{16.47}$$

$$\bar{\delta}_3 = (g_{33})^{-1/2}\delta_3 = (g_{33})^{1/2}\delta^3 = \delta_r$$

The physical components of γ_{ij} are

$$\begin{aligned}
\gamma_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
\gamma_{zz} &= \frac{\partial u_z}{\partial z} \\
\gamma_{rr} &= \frac{\partial u_r}{\partial r} \\
\gamma_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\
\gamma_{\theta r} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) - \frac{u_r}{r} \\
\gamma_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)
\end{aligned} \tag{16.48}$$

and the physical components of γ_i are

$$\begin{aligned}
\gamma_\theta &= \delta_\theta + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \\
\gamma_z &= \delta_z + \frac{\partial u_r}{\partial z} \\
\gamma_r &= \delta_r + \frac{\partial u_r}{\partial r}
\end{aligned} \tag{16.49}$$

Also, the physical components of $\mathcal{K}_{ij}$ are given by

$$\mathcal{K}_{\theta\theta} = \frac{1}{r}\,\frac{\partial\delta_\theta}{\partial\theta} + \frac{\delta_r}{r}$$

$$\mathcal{K}_{zz} = \frac{\partial\delta_z}{\partial z}$$

$$\mathcal{K}_{rr} = \frac{\partial\delta_r}{\partial r}$$

$$\mathcal{K}_{\theta z} = \frac{\partial\delta_\theta}{\partial z} \quad , \quad \mathcal{K}_{z\theta} = \frac{1}{r}\,\frac{\partial\delta_z}{\partial\theta}$$

$$\mathcal{K}_{\theta r} = \frac{\partial\delta_\theta}{\partial r} - \frac{\delta_\theta}{r} \quad , \quad \mathcal{K}_{r\theta} = \frac{1}{r}\,\frac{\partial\delta_r}{\partial\theta} - \frac{\delta_\theta}{r}$$

$$\mathcal{K}_{zr} = \frac{\partial\delta_z}{\partial r} \quad , \quad \mathcal{K}_{rz} = \frac{\partial\delta_r}{\partial z}$$

Next, we note that the physical components of the stress tensor and stress couple tensor may be written as

$$\begin{aligned}
&\tau_{\theta\theta} = r^2\tau^{11} = \frac{1}{r^2}\,\tau_{11} \\
&\tau_{zz} = \tau^{22} = \tau_{22} \\
&\tau_{rr} = \tau^{33} = \tau_{33} \\
&\tau_{\theta z} = r\tau^{12} = \frac{1}{r}\,\tau_{12} \quad , \quad \tau_{z\theta} = r\tau^{21} = \frac{1}{r}\,\tau_{21} \\
&\tau_{\theta r} = r\tau^{13} = \frac{1}{r}\,\tau_{13} \quad , \quad \tau_{r\theta} = r\tau^{31} = \frac{1}{r}\,\tau_{31} \\
&\tau_{zr} = \tau^{23} = \tau_{23} \quad , \quad \tau_{rz} = \tau^{32} = \tau_{32}
\end{aligned} \tag{16.51}$$

$$
\begin{aligned}
s_{\theta\theta} &= r^2 s^{11} = \frac{1}{r^2}\, s_{11} \\
s_{22} &= s^{22} = s_{22} \\
s_{rr} &= s^{33} = s_{33} \\
s_{\theta z} &= r s^{12} = \frac{1}{r}\, s_{12} \quad , \quad s_{z\theta} = r s^{21} = \frac{1}{r}\, s_{21} \\
s_{\theta r} &= r s^{13} = \frac{1}{r}\, s_{13} \quad , \quad s_{r\theta} = r s^{31} = \frac{1}{r}\, s_{31} \\
s_{zr} &= s^{23} = s_{23} \quad , \quad s_{rz} = s^{32} = s_{32}
\end{aligned} \tag{16.52}
$$

Moreover, the physical components of **b**, **c** and **k** are given by

$$b_\theta = r b^1 = \frac{1}{r}\, b_1 \quad , \quad b_z = b^2 = b_2 \quad , \quad b_r = b^3 = b_3 \tag{16.53}$$

$$c_\theta = r c^1 = \frac{1}{r}\, c_1 \quad , \quad c_z = c^2 = c_2 \quad , \quad c_r = c^3 = c_3 \tag{16.54}$$

$$k_\theta = r k^1 = \frac{1}{r}\, k_1 \quad , \quad k_z = k^2 = k_2 \quad , \quad k_r = k^3 = k_3 \tag{16.55}$$

Also, from (.46) and (16.47) we have

$$\ddot{u}_\theta = r\alpha^1 = \frac{1}{r}\, \alpha_1 \quad , \quad \ddot{u}_z = \alpha^2 = \alpha_2 \quad , \quad \ddot{u}_r = \alpha^3 = \alpha_3 \tag{16.56}$$

and

$$\ddot{\delta}_\theta = r\beta^1 = \frac{1}{r}\, \beta_1 \quad , \quad \ddot{\delta}_z = \beta^2 = \beta_2 \quad , \quad \ddot{\delta}_r = \beta^3 = \beta_3 \tag{16.57}$$

where a superposed dot denotes partial differentiation with respect to time. With the help of (16.51) to (16.57) we are able to reduce the equations of motion (16.32) and (16.33) to

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + \rho_o b_r = \rho_o(\ddot{u}_r + y^1\ddot{\delta}_r)$$

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial z} + \frac{2}{r}\,\tau_{\theta r} + \mu_o b_\theta = \rho_o(\ddot{u}_\theta + y^1\ddot{\delta}_\theta) \tag{16.58}$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r} + \rho_o b_z = \rho_o(\ddot{u}_z + y^1\ddot{\delta}_z)$$

and

$$\frac{\partial s_{rr}}{\partial r} + \frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta} + \frac{\partial s_{rz}}{\partial z} + \frac{s_{rr} - s_{\theta\theta}}{r} + (\rho_o c_r - k_r) = \rho_o(y^1\ddot{u}_r + y^2\ddot{\delta}_r)$$

$$\frac{\partial s_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta} + \frac{\partial z_{rz}}{\partial z} + \frac{2}{r}\,s_{\theta r} + (\rho_o c_\theta - k_\theta) = \rho_o(y^1\ddot{u}_\theta + y^2\ddot{\delta}_\theta) \tag{16.59}$$

$$\frac{\partial s_{zr}}{\partial r} + \frac{1}{r}\frac{\partial s_{z\theta}}{\partial \theta} + \frac{\partial s_{zz}}{\partial z} + \frac{s_{zr}}{r} + (\rho_o c_z - k_z) = \rho_o(y^1\ddot{u}_z + y^2\ddot{\delta}_z)$$

where in obtaining (16.58) and (16.59) we have also made use of the expression for covariant differentiation of a second order tensor.

17. Theory of initially spherical composite laminates

In this section we continue to apply the theory of Cosserat composite to initially cylindrical composite laminates.

Consider a composite laminate and let its plies form a set of concentric spherical surfaces. Let x^i (i = 1,2,3) and $\{r,\theta,\phi\}$ denote Cartesian and spherical coordinates with a common origin in an Euclidean three-dimensional space. Let $\mathbf{e}_i$ (i = 1,2,3) and $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_\phi\}$ denote the unit base vectors in the foregoing coordinate systems, respectively. We recall that a spherical surface of radius r may be defined by a position vector of the form

$$\mathbf{P} = r\mathbf{e}_r \tag{17.1}$$

Recalling the relations between the unit base vectors in Cartesian and spherical coordinate systems, i.e.,

$$\begin{aligned} \mathbf{e}_r &= (\sin\phi\cos\theta)\mathbf{e}_1 + (\sin\phi\sin\theta)\mathbf{e}_2 + (\cos\phi)\mathbf{e}_3 \\ \mathbf{e}_\theta &= -(\sin\theta)\mathbf{e}_1 + (\cos\theta)\mathbf{e}_2 \\ \mathbf{e}_\phi &= (\cos\phi\cos\theta)\mathbf{e}_1 + (\cos\phi\sin\theta)\mathbf{e}_2 - (\sin\phi)\mathbf{e}_3 \end{aligned} \tag{17.2}$$

we obtain

$$\mathbf{P} = (r\sin\phi\cos\theta)\mathbf{e}_1 + (r\sin\phi\sin\theta)\mathbf{e}_2 + (r\cos\phi)\mathbf{e}_3 \tag{17.3}$$

Let us now introduce a set of coordinates θ^i (i = 1,2,3) such that

$$\theta^1 = \phi \quad , \quad \theta^2 = \theta \quad , \quad \theta^3 = r \tag{17.4}$$

Hence, in terms of θ^i coordinates we have

$$\mathbf{P} = (\theta^3\sin\theta^1\cos\theta^2)\mathbf{e}_1 + (\theta^3\sin\theta^1\sin\theta^2)\mathbf{e}_2 + (\theta^3\cos\theta^1)\mathbf{e}_3 \tag{17.5}$$

This representation will facilitate much of the intermediate steps especially in connection to cal-

culation of the various quantities of the surface.

In view of the foregoing explanation, we now adopt the following kinematical assumptions for an initially spherical composite laminate

$$\mathbf{R}(r,\theta,\phi) = r\mathbf{e}_r$$

$$\mathbf{D} = \mathbf{A}_3 = \mathbf{e}_r \tag{17.6}$$

$$\mathbf{P}^*(r,\theta,\phi,\zeta) = (r+\zeta)\mathbf{e}_r$$

Making use of (17.4), we can rewrite this

$$\mathbf{R}(\theta^\alpha,\theta^3) = (\theta^3 \sin\theta^1 \cos\theta^2)\mathbf{e}_1 + (\theta^3 \sin\theta^1 \sin\theta^2)\mathbf{e}_2 + (\theta^3 \cos\theta^1)\mathbf{e}_3$$

$$\mathbf{D} = \mathbf{A}_3 = (\sin\theta^1 \cos\theta^2)\mathbf{e}_1 + (\sin\theta^1 \sin\theta^2)\mathbf{e}_2 + \cos\theta^1\, \mathbf{e}_3 \tag{17.7}$$

$$\mathbf{P}^*(\theta^\alpha,\theta^3,\zeta) = [(\theta^3+\zeta)\sin\theta^1 \cos\theta^2]\mathbf{e}_1 + [(\theta^3+\zeta)\sin\theta^1 \sin\theta^2]\mathbf{e}_2 + [(\theta^3+\zeta)\cos\theta^1]\mathbf{e}_3$$

The base vectors of the surface are obtained from $(17.7)_1$ as follows

$$\mathbf{A}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha}$$

Hence we have

$$\mathbf{A}_1 = \mathbf{R}_{,\theta^1} = (\theta^3 \cos\theta^1 \cos\theta^2)\mathbf{e}_1 + (\theta^3 \,\mathrm{cost}\,\theta^1 \sin\theta^2)\mathbf{e}_2 - (\theta^3 \sin\theta^1)\mathbf{e}_3 = \theta^3\mathbf{e}_\phi \tag{17.8}$$

and

$$\mathbf{A}_2 = \mathbf{R}_{,\theta^2} = (-\theta^3 \sin\theta^1 \sin\theta^2)\mathbf{e}_1 + (\theta^3 \sin\theta^1 \cos\theta^2)\mathbf{e}_2 = (\theta^3 \sin\theta^1)\mathbf{e}_\theta \tag{17.9}$$

From (17.8) and (17.9) we obtain the components of the surface metric tensor $A_{\alpha\beta}$

$$A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta$$

Therefore

$$A_{11} = \mathbf{A}_1 \cdot \mathbf{A}_1 = [(\theta^3 \cos\theta^1 \cos\theta^2)]^2 + [(\theta^3 \cos\theta^1 \sin\theta^2)]^2 + [\theta^3 \operatorname{sinc}\theta^1]^2 = (\theta^3)^2$$

$$A_{12} = A_{21} = \mathbf{A}_1 \cdot \mathbf{A}_2 = (\theta^3 \mathbf{e}_\phi) \cdot (\theta^3 \sin\theta^1)\mathbf{e}_\theta = 0 \qquad (17.10.a)$$

$$A_{22} = \mathbf{A}_2 \cdot \mathbf{A}_2 = (\theta^3 \sin\theta^1)^2 \mathbf{e}_\theta \cdot \mathbf{e}_\theta = (\theta^3 \sin\theta^1)^2$$

or

$$(A_{\alpha\beta}) = \begin{bmatrix} (\theta^2)^2 & 0 \\ 0 & (\theta^3 \sin\theta^1)^2 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & (r \sin\theta^1)^2 \end{bmatrix} \qquad (17.10.b)$$

Moreover, we have

$$A^{\alpha\beta} A_{\beta\gamma} = \delta^\alpha{}_\gamma \;\Rightarrow\; A^{\alpha\beta} = (A_{\alpha\beta})^{-1} \qquad (17.11)$$

Hence,

$$A_{\alpha\beta} = \begin{bmatrix} (\theta^3)^2 & 0 \\ 0 & (\theta^3 \sin\theta^1)^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{(\theta^3)^2} & 0 \\ 0 & \frac{1}{(\theta^3 \sin\theta^1)^2} \end{bmatrix} \qquad (17.12)$$

The conjugate base vectors of the surface are given by

$$\mathbf{A}^\alpha = A^{\alpha\beta} \mathbf{A}_\beta$$

Thus,

$$\mathbf{A}^1 = A^{11}\mathbf{A}_1 + A^{12}\mathbf{A}_2 = \frac{\mathbf{A}_1}{(\theta^3)^2} = \frac{1}{\theta^3}\mathbf{e}_\phi$$

$$(17.13)$$

$$\mathbf{A}^2 = A^{21}\mathbf{A}_1 + A^{22}\mathbf{A}_2 = \frac{\mathbf{A}_2}{(\theta^3 \sin\theta^1)^2} = \frac{1}{(\theta^3 \sin\theta^1)}\mathbf{e}_\theta$$

The unit normal to the surface follows from (17.13)

$$A_3 = \frac{A_1 \times A_2}{|A_1 \times A_2|} = \frac{1}{|A_1 \times A_2|} \{(\theta^3 \cos\theta^1 \cos\theta^3)e_1 + (\theta^3 \cos\theta^1 \sin\theta^2)e_2 - (\theta^3 \sin\theta^1)e_3\} \times$$

$$\{-(\theta^3 \sin\theta^1 \sin\theta^2)e_1 + (\theta^3 \sin\theta^1 \cos\theta^2)e_2\}$$

or

$$A_3 = \frac{1}{|A_1 \times A_2|} \{(\theta^3 \cos\theta^2)\cdot(\theta^3 \sin\theta^1 \cos\theta^3)(e_1 \times e_2) - (\theta^3 \cos\theta^1 \sin\theta^2)(\theta^3 \sin\theta^1 \sin\theta^2)(e_2 \times e_1)$$

$$+ (\theta^3 \sin\theta^1)(\theta^3 \sin\theta^1 \sin\theta^2)(e_3 \times e_1) - (\theta^3 \sin\theta^1)(\theta^3 \sin\theta^1 \cos\theta^2)(e_3 \times e_2)\}$$

$$= \frac{1}{|A_1 \times A_2|} (\theta^2)^2 \sin\theta^1 \{(\sin\theta^1 \cos\theta^3)e_1 + (\sin\theta^1 \sin\theta^2)e_2 + (\cos\theta^1)e_3\}$$

$$= (\sin\theta^1 \cos\theta^3)e_1 + (\sin\theta^1 \sin\theta^2)e_2 + (\cos\theta^1)e_3 = e_r \qquad (17.14)$$

We note that A_3 could have been obtained from vector product of e_θ and e_ϕ. However, to illustrate the general procedure we did not make use of e_θ and e_ϕ. The Christoffel symbols of the first and second kind follow from (17.10). The non-vanishing components of the Christoffel symbols are

$$[12,2] = [21,2] = (\theta^3)^2 \sin\theta^1 \cos\theta^1 \quad , \quad [22,1] = -(\theta^3)^2 \sin\theta^1 \cos\theta^1$$

$$\left\{ {2 \atop 12} \right\} = \text{cotan}\,\theta^1 \quad , \quad \left\{ {1 \atop 22} \right\} = \sin\theta^1 \cos\theta^1 \qquad (17.15)$$

and coefficients of the second fundamental form of the surface are given by

$$B_{\alpha\beta} = A_{\alpha,\beta} \cdot A_3 = -A_\alpha \cdot A_{3,\alpha}$$

hence,

$$B_{11} = A_{1,1} \cdot A_3 = [(\theta^3 \cos\theta^1 \cos\theta^2)e_1 + (\theta^3 \cos\theta^1 \sin\theta^2)e_2 - (\theta^3 \sin\theta^1)e_3]_{,1} \cdot$$

$$[(\sin\theta^1 \cos\theta^2)e_1 + (\sin\theta^1 \sin\theta^2)e_2 + (\cos\theta^1)e_3]$$

$$= [-(\theta^3 \sin\theta^1 \cos\theta^2)e_1 - (\theta^3 \sin\theta^1 \sin\theta^2)e_2 - (\theta^3 \cos\theta^1)e_3] \cdot$$

$$[(\sin\theta^1 \cos\theta^2)e_1 + (\sin\theta^1 \sin\theta^2)e_2 + (\cos\theta^1)e_3 = -\theta^3$$

$$B_{12} = A_{2,1} \cdot A_3 = [-(\theta^3 \sin\theta^1 \sin\theta^2)e_1 + (\theta^3 \sin\theta^1 \cos\theta^2)e_2]_{,1} \cdot$$

$$[(\sin\theta^1 \cos\theta^2)e_1 + (\sin\theta^1 \sin\theta^2)e_2 + (\cos\theta^1)e_3] = 0$$

$$B_{21} = A_{1,2} \cdot A_3 = [(\theta^3 \cos\theta^1 \cos\theta^2)e_1 + (\theta^3 \cos\theta^1 \sin\theta^2)e_2 - (\theta^3 \sin\theta^1)e_3]_{,2} \cdot$$

$$[(\sin\theta^1 \cos\theta^2)e_1 + (\sin\theta^1 \sin\theta^2)e_2 + (\cos\theta^1)e_3] = 0$$

$$B_{22} = A_{2,2} \cdot A_3 = [-(\theta^3 \sin\theta^1 \sin\theta^2)e_1 + (\theta^3 \sin\theta^1 \cos\theta^2)e_2]_{,2} \cdot$$

$$[(\sin\theta^1 \cos\theta^2)e_1 + (\sin\theta^1 \sin\theta^2)e_2 + (\cos\theta^1)e_3] = -\theta^3(\sin\theta^1)^2$$

Therefore

$$(B_{\alpha\beta}) = \begin{bmatrix} -\theta^3 & 0 \\ 0 & -\theta^3(\sin\theta^1)^2 \end{bmatrix} = \begin{bmatrix} -r & 0 \\ 0 & -r\sin^2\phi \end{bmatrix} \qquad (17.16)$$

We also have

$$B^\alpha{}_\beta = A^{\alpha\gamma} B_{\gamma\beta}$$

Hence,

$$B^1{}_1 = A^{11}B_{11} + A^{12}B_{21} = -\,\theta^3\,(\frac{1}{\theta^3})^2 = -\,\frac{1}{\theta^3}$$

$$B^1{}_2 = A^{11}B_{21} + A^{12}B_{22} = 0$$

$$B^2{}_1 = A^{21}B_{11} + A^{22}B_{21} = 0$$

$$B^2{}_2 = A^{21}B_{12} + A^{22}B_{22} = -\,(\theta^3 \sin^2 \theta^1)\,\frac{1}{(\theta^3 \sin \theta^1)^2} = -\,\frac{1}{\theta^3}$$

or

$$(B^\alpha{}_\beta) = \begin{bmatrix} -1/\theta^3 & 0 \\ 0 & -1/\theta^3 \end{bmatrix} = \begin{bmatrix} -1/r & 0 \\ 0 & -1/r \end{bmatrix} \tag{17.17}$$

Next, we obtain the various kinematical quantities associated with micro and macro continua for the case of initially spherical composite laminates. From (14.6) and (17.7) it follows that

$$G_1 = \frac{\partial \mathbf{R}}{\partial \theta^1} = (\theta^3 \cos \theta^1 \cos \theta^2)\mathbf{e}_1 + (\theta^3 \cos \theta^1 \sin \theta^2)\mathbf{e}_2 - (\theta^3 \sin \theta^1)\mathbf{e}_3 = r\mathbf{e}_\phi$$

$$G_2 = \frac{\partial \mathbf{R}}{\partial \theta^2} = (-\,\theta^3 \sin \theta^1 \sin \theta^2)\mathbf{e}_1 + (\theta^3 \sin \theta^1 \cos \theta^2)\mathbf{e}_2 = (r \sin \phi)\mathbf{e}_\theta \tag{17.18}$$

$$G_3 = \frac{\partial \mathbf{R}}{\partial \theta^3} = (\sin \theta^1 \cos \theta^2)\mathbf{e}_1 + (\sin \theta^1 \sin \theta^2)\mathbf{e}_2 + (\cos \theta^1)\mathbf{e}_3 = \mathbf{e}_r$$

From (17.18) we obtain

$$G_{11} = G_1 \cdot G_1 = [(\theta^3 \cos \theta^1 \cos \theta^2)\mathbf{e}_1 + (\theta^3 \cos \theta^1 \sin \theta^2)\mathbf{e}_2 + (\theta^3 \sin \theta^1)\mathbf{e}_3] \cdot$$
$$[(\theta^3 \cos \theta^1 \cos \theta^2)\mathbf{e}_1 + (\theta^3 \cos \theta^1 \sin \theta^2)\mathbf{e}_2 - (\theta^3 \sin \theta^1)\mathbf{e}_3] = (\theta^3)^2$$

$$G_{12} = G_{21} = G_1 \cdot G_2 = [(\theta^3 \cos \theta^1 \cos \theta^2)\mathbf{e}_1 + (\theta^3 \cos \theta^1 \sin \theta^2)\mathbf{e}_2 - (\theta^3 \sin \theta^1)\mathbf{e}_3] \cdot$$
$$[(-\theta^3 \sin \theta^1 \sin \theta^2)\mathbf{e}_1 + (\theta^3 \sin \theta^1 \cos \theta^2)\mathbf{e}_2] = 0$$

$$G_{13} = G_{31} = G_1 \cdot G_3 = [(\theta^3 \cos \theta^1 \cos \theta^2)\mathbf{e}_1 + (\theta^3 \cos \theta^1 \sin \theta^2)\mathbf{e}_2 - (\theta^3 \sin \theta^1)\mathbf{e}_3] \cdot$$
$$[(\sin \theta^1 \cos \theta^2)\mathbf{e}_1 + (\sin \theta^1 \sin \theta^2)\mathbf{e}_2 + (\cos \theta^1)\mathbf{e}_3] = 0$$

$$G_{22} = \mathbf{G}_2 \cdot \mathbf{G}_2 = [(-\theta^3 \sin\theta^1 \sin\theta^2)\mathbf{e}_1 + (\theta^3 \sin\theta^1 \cos\theta^2)\mathbf{e}_2] \cdot$$

$$[(-\theta^3 \sin\theta^1 \sin\theta^2)\mathbf{e}_1 + (\theta^3 \sin\theta^1 \cos\theta^2)\mathbf{e}_2] = (\theta^3 \sin\theta^1)^2$$

$$G_{23} = G_{32} = \mathbf{G}_2 \cdot \mathbf{G}_3 = [(-\theta^3 \sin\theta^1 \sin\theta^2)\mathbf{e}_1 + (\theta^3 \sin\theta^1 \cos\theta^2)\mathbf{e}_2] \cdot$$

$$[(\sin\theta^1 \cos\theta^2)\mathbf{e}_1 + (\sin\theta^1 \sin\theta^2)\mathbf{e}_2 + (\cos\theta^1)\mathbf{e}_3] = 0$$

$$G_{33} = \mathbf{G}_3 \cdot \mathbf{G}_3 = [(\sin\theta^1 \cos\theta^2)\mathbf{e}_1 + (\sin\theta^1 \sin\theta^2)\mathbf{e}_2 + (\cos\theta^1)\mathbf{e}_3] \cdot$$

$$[(\sin\theta^1 \cos\theta^2)\mathbf{e}_1 + (\sin\theta^1 \sin\theta^2)\mathbf{e}_2 + (\cos\theta^1)\mathbf{e}_3] = 1$$

Hence

$$(G_{ij}) = \begin{bmatrix} (\theta^3)^2 & 0 & 0 \\ 0 & (\theta^3 \sin\theta^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & (r\sin\phi)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17.19}$$

We also have

$$G^{1/2} = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3$$

$$= \{[(\theta^3 \cos\theta^1 \cos\theta^2)\mathbf{e}_1 + (\theta^3 \cos\theta^1 \sin\theta^2)\mathbf{e}_2 - (\theta^3 \sin\theta^1)\mathbf{e}_3] \times$$

$$[(-\theta^3 \sin\theta^1 \sin\theta^2)\mathbf{e}_1 + (\theta^3 \sin\theta^1 \cos\theta^2)\mathbf{e}_2]\} \cdot$$

$$[(\sin\theta^1 \cos\theta^2)\mathbf{e}_1 + (\sin\theta^1 \sin\theta^2)\mathbf{e}_2 + (\cos\theta^1)\mathbf{e}_3]$$

$$= (\theta^3)^2[(\sin\theta^1 \cos\theta^1)\mathbf{e}_3 + ((\sin\theta^1)^2 \sin\theta^2)\mathbf{e}_2 + ((\sin\theta^1)^2 \cos\theta^2)\mathbf{e}_1] \cdot$$

$$[(\sin\theta^1 \cos\theta^2)\mathbf{e}_1 + (\sin\theta^1 \sin\theta^2)\mathbf{e}_2 + (\cos\theta^1)\mathbf{e}_3]$$

$$= (\theta^3)^2 \sin\theta^1 = r^2 \sin\phi \tag{17.20}$$

Moreover, from (14.9) and (17.7) we have

$$G_1^* = \frac{\partial P^*}{\partial \theta^1} = (\theta^3+\zeta)[(\cos\theta^1 \cos\theta^2)e_1 + (\cos\theta^1 \sin\theta^2)e_2 - (\sin\theta^1)e_3] = (r+\zeta)e_\phi$$

$$G_2^* = \frac{\partial P^*}{\partial \theta^2} = (\theta^3+\zeta)(\sin\theta^1)[-(\sin\theta^2)e_1 + (\cos\theta^2)e_2] = ((r+\zeta)\sin\phi)e_\theta \qquad (17.21)$$

$$G_3^* = \frac{\partial P^*}{\partial \xi} = \frac{\partial P^*}{\partial \zeta} = (\sin\theta^1 \cos\theta^2)e_1 + (\sin\theta^1 \sin\theta^2)e_2 + (\cos\theta^1)e_3 = e_r$$

Also, from (14.10), (14.11) and (17.17) we obtain

$$\nu^\gamma{}_\alpha = \mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \xi B^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \zeta B^\gamma{}_\alpha$$

$$\nu^1{}_1 = 1 - \zeta(-\frac{1}{\theta^3}) = 1 + \frac{\zeta}{r} = \frac{r+\zeta}{r}$$

$$\nu^1{}_2 = \nu^2{}_1 = 0$$

$$\nu^2{}_2 = 1 - \zeta(-\frac{1}{\theta^3}) = 1 + \frac{\zeta}{r} = \frac{r+\zeta}{r}$$

or

$$(\nu^\gamma{}_\alpha) = (\mu^\gamma{}_\alpha) = \begin{bmatrix} \frac{(r+\zeta)}{r} & 0 \\ 0 & \frac{(r+\zeta)}{r} \end{bmatrix} \qquad (17.22)$$

and

$$\nu = D\det(\nu^\gamma{}_\alpha) = \det(\nu^\gamma{}_\alpha) = \mu = (\frac{r+\zeta}{r})^2 \qquad (17.23)$$

Making use of (17.21) we obtain

$$G_{11}^* = G_1^* \cdot G_1^* = [(r+\zeta)e_\phi] \cdot [(r+\zeta)e_\phi] = (r+\zeta)^2$$

$$G_{12}^* = G_{21}^* = G_1^* \cdot G_2^* = [(r+\zeta)e_\phi] \cdot [((r+\zeta)\sin\phi)e_\theta] = 0$$

$$G_{13}^* = G_{31}^* = G_1^* \cdot G_3^* = [(r+\zeta)e_\phi] \cdot [e_r] = 0$$

$$G^*_{22} = G^*_2 \cdot G^*_2 = [((r+\zeta)\sin\phi)^3 e_\theta] \cdot [((r+\zeta)\sin\phi)e_\theta] = ((r+\zeta)\sin\phi)^2$$

$$G^*_{23} = G^*_{32} = G^*_2 \cdot G^*_3 = [((r+\zeta)\sin\phi)e_\theta] \cdot [e_r] = 0$$

$$G^*_{33} = G^*_3 \cdot G^*_3 = (e_r) \cdot (e_r) = 1$$

Hence,

$$(G^*_{ij}) = \begin{bmatrix} (\theta^3+\zeta)^2 & 0 & 0 \\ 0 & ((\theta^3+\zeta)\sin\theta^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (r+\zeta)^2 & 0 & 0 \\ 0 & ((r+\zeta)\sin\phi)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17.24}$$

It then follows that

$$G^{*1/2} = \{\det(G^*_{ij})\}^{1/2} = (\theta^3+\zeta)^2 \sin\theta^1 = (r+\zeta)^2 \sin\phi \tag{17.25}$$

By (17.20) and (17.25) we have

$$(\frac{G^*}{G})^{1/2} = (\frac{r+\zeta}{r})^2 = \nu = \mu \tag{17.26}$$

which confirms (17.23). In view of (17.7) and (17.26) formulae (14.14) and (14.15) reduce to

$$\bar{\tau}^{\alpha 1} = \tau^{\alpha 1} = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha\gamma} v^1{}_\gamma d\zeta = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^3 \tau^{*\alpha 1} d\zeta \tag{17.27.a}$$

$$\bar{\tau}^{\alpha 2} = \tau^{\alpha 2} = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha\gamma} v^2{}_\gamma d\zeta = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^3 \tau^{*\alpha 2} d\zeta \tag{17.27.b}$$

$$\bar{\tau}^{\alpha 3} = \tau^{\alpha 3} = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha 3} d\zeta = v^\alpha \tag{17.27.c}$$

$\bar{\tau}^{3i}$ to be specified by a constitutive relation directly (17.27.d)

$$\bar{s}^{\alpha 1} = s^{\alpha 1} = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha\gamma} v^1{}_\gamma \zeta d\zeta = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^3 \tau^{*\alpha 1} \zeta d\zeta \tag{17.27.e}$$

$$\bar{s}^{\alpha 2} = s^{\alpha 2} = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha\gamma} v^2{}_\gamma \zeta d\zeta = \frac{1}{h_2}\int_o^{h_2} (1 + \frac{\zeta}{r})^3 \tau^{*\alpha 2} \zeta d\zeta \tag{17.27.f}$$

$$\overline{s}^{\alpha 3} = s^{\alpha 3} = \frac{1}{h_2} \int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*\alpha 3} \zeta d\zeta \qquad (17.27.g)$$

$$s^{3i} = 0 \quad \text{or} \quad S^3 = 0 \qquad (17.27.h)$$

$$k^1 = \frac{1}{h_2} \int_o^{h_2} (1 + \frac{\zeta}{r})^3 \tau^{*31} d\zeta \ , \ k^2 = \frac{1}{h_2} \int_o^{h_2} (1 + \frac{\zeta}{r})^3 \tau^{*32} d\zeta \qquad (17.27.i)$$

$$k^3 = \frac{1}{h_2} \int_o^{h_2} (1 + \frac{\zeta}{r})^2 \tau^{*33} d\zeta \qquad (17.27.j)$$

$$v^3 = \frac{1}{h_2} \int_o^{h_2} (1 + \frac{\zeta}{r})^2 [\tau^{*33} - (1 + \frac{\zeta}{r}) \tau^{*11} \frac{\zeta}{r}] d\zeta \qquad (17.27.k)$$

It is interesting to observe that when the radius of the spherical laminate becomes large (i.e., when the cylindrical surface approaches a flat surface) the value of $\frac{\zeta}{r}$ becomes small and may be neglected in comparison to unity (ideally $\frac{\zeta}{r}$ approaches zero) and the various expression obtained in this section will reduce to those obtained for an initially flat composte laminate.

The relative kinematical measures γ_{ij}, γ_i and $\mathcal{K}_{ij}$ are now given by

$$\gamma_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i}) \qquad (17.28)$$

$$\gamma_i = \delta_i + u_{3|i} \qquad (17.29)$$

$$\mathcal{K}_{ij} = \delta_{i|j} \qquad (17.30)$$

where a vertical bar (|) denotes covariant differentiation with respect to coordinates θ^i (i = 1,2,3) as specified by (17.4). Moreover, equations of motion are given by

$$\tau_j^i{}_{|i} + \rho_o b_j = \rho_o(\alpha_j + y^1 \beta_j) \qquad (17.31)$$

$$s_j^i{}_{|i} + (\rho_o c_j - k_j) = \rho_o(y^1 \alpha_j + y^2 \beta_j) \qquad (17.32)$$

where all components in the above are referred to coordinates θ^i (i = 1,2,3).

For convenience and systematic reduction of various results of this section we adopted the coordinate system (17.4). However, most of the available results in continuum mechanics regarding spherical bodies are in terms of the spherical coordinates r,θ,ϕ. In order to write the relevant results of this section in terms of r,θ,ϕ we consider the representation (17.6) and adopt a system of cylindrical coordinates r,θ,ϕ such that

$$\theta^1 = \phi \quad , \quad \theta^2 = \theta \quad , \quad \theta^3 = r \tag{17.33}$$

From $(17.6)_1$ and (17.33) it follows

$$\mathbf{G}_1 = r\mathbf{e}_\phi \quad , \quad \mathbf{G}_2 = (r \sin \phi)\mathbf{e}_\theta \quad , \quad \mathbf{G}_3 = \mathbf{e}_r \tag{17.34}$$

and

$$(G_{ij}) = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & (r \sin \phi)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (G^{ij}) = \begin{bmatrix} 1/r^2 & 0 & 0 \\ 0 & 1/(r \sin \phi)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17.35}$$

$$G^{1/2} = r^2 \sin \phi \tag{17.36}$$

Moreover, from $(17.6)_3$ we obtain

$$\mathbf{G}_1^* = (r+\zeta)\mathbf{e}_\phi \quad , \quad \mathbf{G}_2 = ((r+\zeta)\sin \phi)\mathbf{e}_\theta \quad , \quad \mathbf{G}_3 = \mathbf{e}_r \tag{17.37}$$

and

$$(G_{ij}^*) = \begin{bmatrix} (r+\zeta)^2 & 0 & 0 \\ 0 & ((r+\zeta)\sin \phi)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad (G^{*ij}) = \begin{bmatrix} 1/(r+\zeta)^2 & 0 & 0 \\ 0 & 1/((r+\zeta)\sin \phi)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17.38}$$

$$G^{*1/2} = (r+\zeta)^2 \sin \phi \tag{17.39}$$

From (14.13), (17.36) and (17.39) it follows

$$\nu = \mu = \left(\frac{G^*}{G}\right)^{1/2} = \left(\frac{r+\zeta}{r}\right)^2 = \left(1 + \frac{\zeta}{r}\right)^2 \tag{17.40}$$

as before. In order to calculate expresisons involving covariant differentiation we need to calculate the Christoffel symbols of the first and second kind. Christoffel symbols of the first kind are given by

$$[ijk] = \frac{1}{2}\,(g_{jk,i} + g_{ki,j} - g_{ij,k}) \tag{17.41}$$

The only non-vanishing Christoffel symbols of the first kind are

$$\begin{gathered} [11,2] = -r \quad , \quad [22,1] = -r^2 \sin\phi \cos\phi \quad , \quad [22,3] = -r \sin^2\phi \\ [12,2] = [21,2] = r^2 \sin\phi \cos\phi \quad , \quad [13,1] = [31,1] = r \quad , \quad [23,2] = [32,2] = r \sin^2\phi \end{gathered} \tag{17.42}$$

Christoffel symbols of the second kind are given by

$$\{{}_i{}^k{}_j\} = g^{km}[ijm] \tag{17.43}$$

From (17.42) and (17.43) the only non-vanishing Christoffel symbols of the second kind are

$$\begin{gathered} \{{}_1{}^3{}_1\} = -r \quad , \quad \{{}_2{}^1{}_2\} = -\sin\phi\cos\phi \quad , \quad \{{}_2{}^3{}_2\} = -r\sin^2\phi \\ \{{}_1{}^2{}_2\} = \{{}_2{}^2{}_1\} = \cot\text{an}\,\phi \quad , \quad \{{}_1{}^1{}_3\} = \{{}_3{}^1{}_1\} = \frac{1}{r} \quad , \quad \{{}_2{}^2{}_3\} = \{{}_3{}^2{}_2\} = \frac{1}{r} \end{gathered} \tag{17.44}$$

The physical components of the displacement vector **u** and the director displacement δ are given by

$$\begin{aligned} \bar{u}_1 &= (g_{11})^{-1/2}u_1 = (g_{11})^{1/2}u^1 = u_\phi \\ \bar{u}_2 &= (g_{22})^{-1/2}u_2 = (g_{22})^{1/2}u^2 = u_\theta \\ \bar{u}_3 &= (g_{33})^{-1/2}u_3 = (g_{33})^{1/2}u^3 = u_r \end{aligned} \tag{17.45}$$

$$\bar{\delta}_1 = (g_{11})^{-1/2}\delta_1 = (g_{11})^{1/2}\delta^1 = \delta_\phi$$

$$\bar{\delta}_2 = (g_{22})^{-1/2}\delta_2 = (g_{22})^{1/2}\delta^2 = \delta_\theta \qquad (17.46)$$

$$\bar{\delta}_3 = (g_{33})^{-1/2}\delta_3 = (g_{33})^{1/2}\delta^3 = \delta_r$$

The physical components of γ_{ij} are

$$\gamma_{\phi\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}u_r$$

$$\gamma_{\theta\theta} = \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} + \frac{1}{r}(\text{cotan}\,\phi)u_\phi$$

$$\gamma_{rr} = \frac{\partial u_r}{\partial r}$$

$$(17.47)$$

$$\gamma_{\phi\theta} = \frac{1}{2}\left(\frac{1}{r\sin\phi}\frac{\partial u_\phi}{\partial\theta} + \frac{1}{r}\frac{\partial u_\theta}{\partial\phi} - \frac{\text{cotan}\,\phi}{r}u_\theta\right)$$

$$\gamma_{r\phi} = \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\phi} - \frac{1}{r}u_\phi\right)$$

$$\gamma_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\theta} - \frac{1}{r}u_\theta\right)$$

and the physical components of γ_i are

$$\gamma_\phi = \delta_\phi + \frac{1}{r}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r}$$

$$\gamma_\theta = \delta_\theta + \frac{1}{r\sin\phi}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} \qquad (17.48)$$

$$\gamma_r = \delta_r + \frac{\partial u_r}{\partial r}$$

Also, the physical components of $\mathcal{K}_{ij}$ are given by

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$$\mathcal{K}_{\phi\phi} = \frac{1}{r}\,\frac{\partial \delta_\phi}{\partial \phi} + \frac{\delta_r}{r} + \frac{1}{r^2}\,\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r^2}\,u_r$$

$$\mathcal{K}_{\theta\theta} = \frac{1}{r \sin\phi}\,\frac{\partial \delta_\theta}{\partial \theta} + \frac{1}{r}\,\mathrm{cotan}\,\phi\,\delta_\phi + \frac{\delta_r}{r} + \frac{1}{r^2 \sin\phi}\,\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2}\,\mathrm{cotan}\,\phi\,u_\phi + \frac{u_r}{r^2}$$

$$\mathcal{K}_{rr} = \partial \delta_r \mathrm{ovr} \partial r$$

$$\mathcal{K}_{\phi\theta} = \frac{1}{r \sin\phi}\,\frac{\partial \delta_\phi}{\partial \theta} - \frac{1}{r}\,\mathrm{cotan}\,\phi\,\delta_\theta + \frac{1}{r^2}\,\frac{\partial u_\theta}{\partial \phi} - \frac{1}{r^2}\,\mathrm{cotan}\,\phi\,u_\phi$$

$$\mathcal{K}_{\theta\phi} = \frac{1}{r}\,\frac{\partial \delta_\theta}{\partial \phi} - \frac{1}{r^2}\,\mathrm{cotan}\,\phi\,u_\theta + \frac{1}{r^2 \sin\phi}\,\frac{\mathrm{partualu}_\phi}{\partial \theta} \tag{17.49}$$

$$\mathcal{K}_{\phi r} = \frac{\partial \delta_\phi}{\partial r}$$

$$\mathcal{K}_{r\phi} = \frac{1}{r}\,\frac{\partial \delta_r}{\partial \phi} - \frac{1}{r}\,\delta_\phi + \frac{1}{r}\,\frac{\partial u_\phi}{\partial r}$$

$$\mathcal{K}_{\theta r} = \frac{\partial \delta_\theta}{\partial r}$$

$$\mathcal{K}_{r\theta} = \frac{1}{r \sin\phi}\,\frac{\partial \delta_r}{\partial \theta} - \frac{1}{r}\,\delta_\theta + \frac{1}{r}\,\frac{\partial u_\theta}{\partial r}$$

Next, we note that the physical components of the stress tensor and stress couple tensor may be written as

$$\tau_{\phi\phi} = r^2\tau^{11} = \frac{1}{r^2}\,\tau_{11}$$

$$\tau_{\theta\theta} = (r \sin \phi)^2\tau^{22} = \frac{1}{(r \sin \phi)^2}\,\tau_{22}$$

$$\tau_{rr} = \tau^{33} = \tau_{33}$$

$$t_{\phi\theta} = (r^2 \sin \phi) = \frac{1}{r^2 \sin \phi}\,\tau_{12} \quad , \quad \tau_{\theta\phi} = (r^2 \sin \phi)\tau^{21} = \frac{1}{r^2 \sin \phi}\tau_{21} \tag{17.50}$$

$$\tau_{\phi r} = r\tau^{13} = \frac{1}{r}\,\tau_{13} \quad , \quad \tau_{r\phi} = r\tau^{31} = \frac{1}{r}\,\tau_{31}$$

$$\tau_{\theta r} = (r \sin \phi)\tau^{23} = \frac{1}{r \sin \phi}\,\tau_{23} \quad , \quad \tau_{r\theta} = (r \sin \phi)\tau^{32} = \frac{1}{r \sin \phi}\,\tau_{32}$$

$$s_{\phi\phi} = r^2 s^{11} = \frac{1}{r^2}\,s_{11}$$

$$s_{\theta\theta} = (r \sin \phi)^2\, s^{22} = \frac{1}{(r \sin \phi)^2}\,s_{22}$$

$$s_{rr} = s^{33} = s_{33}$$

$$s_{\phi\theta} = (r^2 \sin \phi)s^{12} = \frac{1}{r^2 \sin \phi}\,s_{12} \quad , \quad s_{\theta\phi} = (r^2 \sin \phi)s^{21} = \frac{1}{(r^2 \sin \phi)}\,s_{21} \tag{17.51}$$

$$s_{\phi r} = r s^{13} = \frac{1}{r}\,s_{13} \quad , \quad s_{r\phi} = r s^{31} = \frac{1}{r}\,s_{31}$$

$$s_{\theta r} = (r \sin \phi)s^{23} = \frac{1}{r \sin \phi}\,s_{23} \quad , \quad s_{r\theta} = (r \sin \phi)s^{32} = \frac{1}{r \sin \phi}\,s_{32}$$

Moreover, the physical components of **b**, **c** and **k** are given by

$$b_\phi = rb^1 = \frac{1}{r}\,b_1 \quad , \quad b_\theta = (r \sin \phi)b^2 = \frac{1}{r \sin \phi}\,b_2 \quad , \quad b_r = b^3 = b_3 \tag{17.52}$$

$$c_\phi = rc^1 = \frac{1}{r}\,c_1 \quad , \quad c_\theta = (r \sin \phi)b^2 = \frac{1}{r \sin \phi}\,b_2 \quad , \quad c_r = c^3 = c_3 \tag{17.53}$$

$$k_\phi = rk^1 = \frac{1}{r}\,k_1 \quad , \quad k_\theta = (r \sin \phi)k^2 = \frac{1}{r \sin \phi}\,k_2 \quad , \quad k_r = k^3 = k_3 \tag{17.54}$$

Also, from (17.45) and (17.46) we have

$$\ddot{u}_\phi = r\alpha^1 = \frac{1}{r}\,\alpha_1 \quad , \quad \ddot{u}_\theta = (r\sin\phi)\alpha^1 = \frac{1}{r\sin\phi}\,\alpha_2 \quad , \quad \ddot{u}_r = \alpha^3 = \alpha_3 \tag{17.55}$$

and

$$\ddot{\delta}_\phi = r\beta^1 = \frac{1}{r}\,\beta_1 \quad , \quad \ddot{\delta}_\theta = (r\sin\phi)\beta^2 = \frac{1}{r\sin\phi}\,\beta_2 \quad , \quad \ddot{\delta}_r = \beta^3 = \beta_3 \tag{17.56}$$

where a superposed dot denotes partial differentiation with respect to time. With the help of (17.50) to (17.56) we are able to reduce the equations of motion (17.31) and (17.32) to

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r\sin\phi}\,\frac{\partial\tau_{\theta r\theta}}{\partial\theta} + \frac{1}{r}\,\frac{\partial\tau_{\phi r\phi}}{\partial\phi} + \frac{2}{r}\,\tau_{rr} + \frac{\text{cotan}\,\phi}{r}\,\tau_{\phi r} - \frac{1}{r}\,(\tau_{\theta\theta} + \tau_{\phi\phi}) + \rho_o b_r = \rho_o(\ddot{u}_r + y^1\ddot{\delta}_r)$$

$$\frac{\partial\tau_{\theta r}}{\partial r} + \frac{1}{r\sin\phi}\,\partial\frac{\tau_{\theta\theta}}{\partial\theta} + \frac{1}{r}\,\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{1}{r}\,[3\tau_{\theta r} - 2(\text{cotan}\,\phi)\tau_{\theta\phi}] + \rho_o b_\theta = \rho_o(\ddot{u}_\theta + y^1\ddot{\delta}_\theta) \tag{17.57}$$

$$\frac{\partial\tau_{\phi r}}{\partial r} + \frac{1}{r\sin\theta}\,\frac{\partial\tau_{\phi\theta}}{\partial\theta} + \frac{1}{r}\,\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{1}{r}\,[3\tau_{\phi r} + (\tau_{\phi\phi} - \tau_{\theta\theta})] + \rho_o b_z = \rho_o(\ddot{u}_z + y^1\ddot{\delta}_z)$$

and

$$\frac{\partial s_{rr}}{\partial r} + \frac{1}{r\sin\phi}\,\frac{\partial s_{r\theta}}{\partial\theta} + \frac{1}{r}\,\frac{\partial s_{r\phi}}{\partial\phi} + \frac{2}{r}\,s_{rr} + \frac{\text{cotan}\,\phi}{r}\,s_{r\phi} - \frac{1}{r}\,(s_{\theta\theta} + s_{\phi\phi}) + (\rho_o c_r - k_r) = \rho_o(y^1\ddot{u}_r + y^2\ddot{\delta}_r)$$

$$\frac{\partial s_{\theta r}}{\partial r} + \frac{1}{r\sin\phi}\,\frac{\partial s_{\theta\theta}}{\partial\theta} + \frac{1}{r}\,\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{1}{r}\,[3s_{\theta r} - 2(\text{cotan}\,\phi)s_{\theta\phi}] + (\rho_o c_\theta - k_\theta) = \rho_o(y^1\ddot{u}_\theta + y^2\ddot{\delta}_\theta) \tag{17.58}$$

$$\frac{\partial s_{\phi r}}{\partial r} + \frac{1}{r\sin\theta}\,\frac{\partial s_{\phi\theta}}{\partial\theta} + \frac{1}{r}\,\frac{\partial_{\phi\phi}}{\partial\phi} + \frac{1}{r}\,[3s_{\phi r} + (s_{\phi\phi} - s_{\theta\theta})] + (\rho_o c_z - k_z) = \rho_o(y^1\ddot{u}_z + y^2\ddot{\delta}_z)$$

where in obtaining (17.57) and (17.58) we have also made use of the expression for covariant differentiation of a second order tensor.

18. Constitutive coefficients for an initially flat composite laminate in bending

Using the procedure of Section (13), in this section we obtain explicit forms for constitutive equations for linear theory of composite laminates with initially flat thin plies of uniform thickness. Here we confine our attention to composite laminate composed of alternating layers of only two elastic materials, each of which is assumed to be homogeneous and isotropic. It should be mentined that any derivation of constitutive equations from three-dimensional theory involves some approximations and special assumptions.

Prior to the calculation of an explicit form for the constitutive equation we need to dispose of certain preliminaries. To this end we consider a composite laminate and assume that the position vector $\mathbf{P}^*$, of the micro-body $\mathcal{B}^*$, in a reference configuration is given by

$$\mathbf{P}^* = \mathrm{R}(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3) \tag{18.1}$$

We recall that in general $\mathbf{D}$ is a three-dimensional vector having components D^1,D^2,D^3 in the direction of G_1,G_2,G_3. However, in the reference configuration without loss of generality we may specify $\mathbf{D}$ by

$$\mathbf{D} = \mathrm{D}\mathbf{A}_3 \quad , \quad D_\alpha = 0 \quad , \quad D_3 = D(\eta^\alpha,\theta^3) \tag{18.2}$$

where $\mathbf{A}_3 = \mathbf{A}_3(\eta^\alpha)$ is the unit normal to the Cosserat surface, i.e., shell-like micro-structure at composite particle P. Here we make further assumption that the position vector $\mathbf{R}$ and the director $\mathbf{D}$, in the reference configuration are given by

$$\mathbf{R}(\theta^\alpha,\theta^3) = \bar{\mathrm{R}}(\theta^\alpha) + \theta^3\mathbf{A}_3 \tag{18.3}$$

$$\mathbf{D} = \mathbf{A}_3 \quad , \quad D_\alpha = 0 \quad , \quad D_3 = D = 1 \tag{18.4}$$

In view of $(18.4)_3$ we have

$$\xi = \zeta \tag{18.5}$$

in the reference configuration. Hence (18.1) may be written as

$$P^* = R(\eta^\alpha, \theta^3) + \zeta A_3(\eta^\alpha, \theta^3) \tag{18.6}$$

The base vectors G_i^* are obtained from (18.6) and are given by

$$\begin{gathered} G_\alpha^* = R_{,\alpha} + \zeta A_{3,\alpha} = A_\alpha + \zeta A_{3,\alpha} \\ G_3^* = A_3 \end{gathered} \tag{18.7}$$

Since $A_{3,\alpha} = -B^\gamma{}_\alpha A_\gamma$ we may write (18.7) as follows:

$$\begin{gathered} G_\alpha^* = A_\alpha + \zeta(-B^\gamma{}_\alpha)A_\gamma = \delta^\gamma{}_\alpha A_\gamma - \zeta B^\gamma{}_\alpha A_\gamma = \mu^\gamma{}_\alpha A_\gamma \\ G_3^* = A_3 \end{gathered} \tag{18.8}$$

where

$$\mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \zeta B^\gamma{}_\alpha \tag{18.9}$$

In the case of a flat plate $B^\gamma{}_\alpha = 0$ and (18.8) reduce to

$$G_\alpha^* = A_\alpha \quad , \quad G_3^* = A_3 \tag{18.10}$$

we also obtain in this case

$$\begin{gathered} G_{\alpha\beta}^* = G_\alpha^* \cdot G_\alpha^* = A_\alpha \cdot A_\beta = A_{\alpha\beta} \\ G_{\alpha 3}^* = G_\alpha^* \cdot A_3 = A_\alpha \cdot A_3 = 0 \\ G_{33}^* = G_3^* \cdot G_3^* = A_3 \cdot A_3 = 1 \end{gathered} \tag{18.11}$$

Recall the Gibbs function ϕ^* for an initially homogeneous and isotropic elastic material, i.e.,

$$\rho_o^* \phi^* = \{ -\frac{1+\nu^*}{2E^*} G_{im}^* G_{jn}^* + \frac{\nu^*}{2E^*} G_{ij}^* G_{mn}^* \} \tau^{*ij} \tau^{*mn} \tag{18.12}$$

BASE

We now expand each term on the right-hand side of (18.12). Considering the second term, we may write

$$G^*_{ij}G^*_{mn}\tau^{*ij}\tau^{*mn} = G^*_{\alpha j}G^*_{mn}\tau^{*\alpha j}\tau^{*mn} + G^*_{3j}G^*_{mn}\tau^{*3j}\tau^{*mn}$$

$$= G^*_{\alpha\beta}G^*_{mn}\tau^{*\alpha\beta}\tau^{*MN} + G^*_{\alpha 3}G^*_{mn}\tau^{*\alpha 3}\tau^{*mn}$$

$$+ G^*_{3\beta}G^*_{mn}\tau^{*3\beta}\tau^{*mn} + G^*_{33}G^*_{mn}\tau^{*33}\tau^{*mn}$$

$$= G^*_{\alpha\beta}G^*_{mn}\tau^{*\alpha\beta}\tau^{*mn} + G^*_{\alpha\beta}G^*_{3n}\tau^{*33}\tau^{*mn}$$

$$= G^*_{\alpha\beta}G^*_{\gamma n}\tau^{*\alpha\beta}\tau^{*\gamma n} + G^*_{\alpha\beta}G^*_{3n}\tau^{*\alpha\beta}\tau^{*3n}$$

$$+ G^*_{33}G^*_{\gamma n}\tau^{*33}\tau^{*\gamma n} + G^*_{33}G^*_{3n}\tau^{*33}\tau^{*3n}$$

$$= G^*_{\alpha\beta}G^*_{\gamma\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + G^*_{\alpha\beta}G^*_{\gamma 3}\tau^{*\alpha\beta}\tau^{*\gamma 3}$$

$$+ G^*_{\alpha\beta}G^*_{3\delta}\tau^{*\alpha\beta}\tau^{*3\delta} + G^*_{\alpha\beta}G^*_{33}\tau^{*\alpha\beta}\tau^{*33}$$

$$+ G^*_{33}G^*_{\gamma\delta}\tau^{*33}\tau^{*\gamma\delta} + G^*_{33}G^*_{\gamma 3}\tau^{*33}\tau^{*\gamma 3}$$

$$+ G^*_{33}G^*_{3\gamma}\tau^{*33}\tau^{*3\delta} + G^*_{33}G^*_{33}\tau^{*33}\tau^{*33}$$

or

$$G^*_{ij}G^*_{mn}\tau^{*ij}\tau^{*mn} = G^*_{\alpha\beta}G^*_{\gamma\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + G^*_{\alpha\beta}G^*_{33}\tau^{*\alpha\beta}\tau^{*33}$$

$$+ G^*_{33}G^*_{\gamma\delta}\tau^{*33}\tau^{*\gamma\delta} + (G^*_{33})^2(\tau^{*33})^2$$

or

$$G^*_{ij}G^*_{mn}\tau^{*ij}\tau^{*mn} = G^*_{\alpha\beta}G^*_{\gamma\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + 2G^*_{\alpha\beta}\tau^{*\alpha\beta}\tau^{*33} + (\tau^{*33})^2 \qquad (18.13)$$

where in obtaining (18.13) we have made use of $(18.11)_{2,3}$. Substituting $(18.11)_1$ into (18.13) we may rewrite (18.13) as follows:

$$G^*_{ij}G^*_{mn}\tau^{*ij}\tau^{*mn} = A_{\alpha\beta}A_{\gamma\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + 2A_{\alpha\beta}\tau^{*\alpha\beta}\tau^{*33} + (\tau^{*33})^2 \tag{18.14}$$

Considering the first term on the right-hand side of (18.12), we may write:

$$\begin{aligned}
G^*_{im}G^*_{jn}\tau^{*ij}\tau^{*mn} &= G^*_{\alpha m}G^*_{jn}\tau^{*\alpha j}\tau^{*mn} + G^*_{3m}G^*_{jn}\tau^{*3j}\tau^{*mn} \\
&= G^*_{\alpha\gamma}G^*_{jn}\tau^{*\alpha j}\tau^{*\gamma n} + G^*_{\alpha 3}G^*_{jn}\tau^{*\alpha j}\tau^{*3n} \\
&\quad + G^*_{3\gamma}G^*_{jn}\tau^{*3j}\tau^{*\gamma n} + G^*_{33}G^*_{jn}\tau^{*3j}\tau^{*3n} \\
&= G^*_{\alpha\gamma}G^*_{jn}\tau^{*\alpha j}\tau^{*\gamma n} + G^*_{33}G^*_{jn}\tau^{*3j}\tau^{*3n} \\
&= G^*_{\alpha\gamma}G^*_{\beta n}\tau^{*\alpha\beta}\tau^{*\gamma n} + G^*_{\alpha\gamma}G^*_{3n}\tau^{*\alpha 3}\tau^{*\gamma n} \\
&\quad + G^*_{33}G^*_{\beta n}\tau^{*3\beta}\tau^{*3n} + G^*_{33}G^*_{3n}\tau^{*33}\tau^{*3n} \\
&= G^*_{\alpha\gamma}G^*_{\beta\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + G^*_{\alpha\gamma}G^*_{\beta 3}\tau^{*\alpha\beta}\tau^{*\gamma 3} \\
&\quad + G^*_{\alpha\gamma}G^*_{3\delta}\tau^{*\alpha 3}\tau^{*\gamma\delta} + G^*_{\alpha\gamma}G^*_{33}\tau^{*\alpha 3}\tau^{*\gamma 3} \\
&\quad + G^*_{33}G^*_{\beta\delta}\tau^{*3\beta}\tau^{*3\delta} + G^*_{33}G^*_{\beta 3}\tau^{*3\beta}\tau^{*33} \\
&\quad + G^*_{33}G^*_{3\delta}\tau^{*33}\tau^{*3\delta} + G^*_{33}G^*_{33}\tau^{*33}\tau^{*33}
\end{aligned}$$

or

$$\begin{aligned}
G^*_{im}G^*_{jn}\tau^{*ij}\tau^{*mn} &= G^*_{\alpha\gamma}G^*_{\beta\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + G^*_{\alpha\gamma}G^*_{33}\tau^{*\alpha 3}\tau^{*\gamma 3} \\
&\quad + G^*_{33}G^*_{\beta\delta}\tau^{*3\beta}\tau^{*3\delta} + (G^*_{33})^2(\tau^{*33})^2
\end{aligned}$$

or

$$G^*_{im}G^*_{jn}\tau^{*ij}\tau^{*mn} = G^*_{\alpha\gamma}G^*_{\beta\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + 2G^*_{\alpha\gamma}\tau^{*\alpha 3}\tau^{*\gamma 3} + (\tau^{*33})^2 \tag{18.15}$$

where in obtaining (18.15) we have made use of $(18.11)_{2,3}$ and the symmetry of stress tensor τ^{*ij}. Substituting $(18.11)_1$ into (18.15), we may rewrite (18.15) as follows:

$$G^*_{im}G^*_{jn}\tau^{*ij}\tau^{*mn} = A_{\alpha\gamma}A_{\beta\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + 2A_{\alpha\beta}\tau^{*\alpha 3}\tau^{*\beta 3} + (\tau^{*33})^2 \tag{18.16}$$

Substituting (18.14) and (18.16) into (18.12), we obtain

$$\rho_o^*\phi^* = -\frac{1+\nu^*}{2E^*}\{A_{\alpha\gamma}A_{\beta\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + 2A_{\alpha\beta}\tau^{*\alpha 3}\tau^{*\beta 3} + (\tau^{*33})^2\}$$

$$+\frac{\nu^*}{2E^*}\{A_{\alpha\beta}A_{\gamma\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + 2A_{\alpha\beta}\tau^{*\alpha\beta}\tau^{*33} + (\tau^{*33})^2\}$$

or

$$\rho_o^*\phi^* = \frac{1}{2E^*}\{\nu^* A_{\alpha\beta}A_{\gamma\delta} - (1+\nu^*)A_{\alpha\gamma}A_{\beta\delta}\}\tau^{*\alpha\beta}\tau^{*\gamma\delta}$$

$$+\frac{1}{E^*}A_{\alpha\beta}\{\nu^*\tau^{*\alpha\beta}\tau^{*33} - (1+\nu^*)\tau^{*\alpha 3}\tau^{*\beta 3}\} - \frac{1}{2E^*}(\tau^{*33})^2 \tag{18.17}$$

At this point we introduce five coefficients q_1, q_2, q_3, q_4 and q_5 and rewrite (18.17) as follows

$$\rho_o^*\phi^* = \frac{1}{2E^*}\{q_1\nu^* A_{\alpha\beta}A_{\gamma\delta} - q_2(1+\nu^*)A_{\alpha\gamma}A_{\beta\delta}\}\tau^{*\alpha\beta}\tau^{*\gamma\delta}$$

$$+\frac{1}{E^*}A_{\alpha\beta}\{q_3\nu^*\tau^{*\alpha\beta}\tau^{*33} - q_4(1+\nu^*)\tau^{*\alpha 3}\tau^{*\beta 3}\} - \frac{1}{2E^*}q_5(\tau^{*33})^2 \tag{18.18}$$

Coefficients q_1 to q_5 are assumed to be constants and may be used as tracers for each term or calibrating factors if such need arises. Otherwise these coefficients can be put equal to unity. We call these "calibration coefficients."

Introducing (18.18) into (13.4) and recalling that for an initially flat plate we have $\mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha$ we obtain

$$\rho_o(h_1+h_2)\phi = \int_o^{h_1+h_2} \{ \frac{1}{2E^*} [q_1\nu^* A_{\alpha\beta}A_{\gamma\delta} - q_2(1+\nu^*)A_{\alpha\gamma}A_{\beta\delta}]\tau^{*\alpha\beta}\tau^{*\gamma\delta}$$

$$+ \frac{1}{E^*} A_{\alpha\beta}[q_3\nu^*\tau^{*\alpha\beta}\tau^{*33} - q_4(1+\nu^*)\tau^{*\alpha 3}\tau^{*\beta 3}] - \frac{1}{2E^*} q_5(\tau^{*33})^2\} d\zeta$$

$$= q_1 \int_o^{h_1+h_2} \frac{\nu^*}{2E^*} A_{\alpha\beta}A_{\gamma\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta - q_2 \int_o^{h_1+h_2} \frac{(1+\nu^*)}{2E^*} A_{\alpha\gamma}A_{\beta\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta$$

$$+ q_3 \int_o^{h_1+h_2} \frac{\nu^*}{E^*} A_{\alpha\beta}\tau^{*\alpha\beta}\tau^{*33} d\zeta - q_4 \int_o^{h_1+h_2} \frac{1+\nu^*}{E^*} A_{\alpha\beta}\tau^{*\alpha 3}\tau^{*\beta 3} d\zeta$$

$$- q_5 \int_o^{h_1+h_2} \frac{1}{2E^*} (\tau^{*33})^2 d\zeta \qquad (18.19)$$

Since $A_{\alpha\beta}$ are independent from ζ, we may further reduce (18.19) and write

$$\rho_o(h_1+h_2)\phi = q_1 A_{\alpha\beta}A_{\gamma\delta}(\int_o^{h_1} \frac{\nu_1^*}{2E_1^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta + \int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{2E_2^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta)$$

$$- q_2 A_{\alpha\gamma}A_{\beta\delta} (\int_o^{h_1} \frac{(1+\nu_1^*)}{2E_1^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta + \int_{h_1}^{h_1+h_2} \frac{(1+\nu_2^*)}{2E_2^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta)$$

$$+ q_3 A_{\alpha\beta}(\int_o^{h_1} \frac{\nu_1^*}{E_1^*} \tau^{*\alpha\beta}\tau^{*33} d\zeta + \int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{e_2^*} \tau^{*\alpha\beta}\tau^{*33} d\zeta)$$

$$- q_4 A_{\alpha\beta}(\int_o^{h_1} \frac{1+\nu_1^*}{E_1^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta + \int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{E_2^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta)$$

$$- q_5(\int_o^{h_1} \frac{1}{2E_1^*} (\tau^{*33})^2 d\zeta + \int_{h_1}^{h_1+h_2} \frac{1}{2E_2^*} (\tau^{*33})^2 d\zeta) \qquad (18.20)$$

In order to calculate an expression for ϕ in the form (18.20), we need to introduce suitable assumptions for the stresses τ^{*ij} in terms of composite stress τ^{ij}, composite couple stress s^{ij} and composite intrinsic force k^i defined previously. Also, in the calculation of ϕ, we shall assume that the effect of the interlaminar stresses in the constitutive relations is negligible.

Since the two-dimensional equations governing the behavior of the micro-structure separate into those for bending and extensional cases, it is more convenient to carry out the calculation for ϕ in two parts. Thus, employing again the same symbol for a function and its value, we write

$$\phi = \phi_b + \phi_e \tag{18.21}$$

where ϕ_b and ϕ_e are associated with the bending and extension cases, respectively. We therefore proceed to calculate the expressions for $\rho_o\phi_b$ and $\rho_o\phi_e$.

Considering the case of bending, we introduce the following expressions for stresses

$$\tau^{*\alpha\beta} = \left(\frac{6s^{\alpha\beta}}{(h_1+h_2)}\,\frac{\zeta-h_1}{(h_1+h_2)/2}\right) = \frac{12s^{\alpha\beta}}{(h_1+h_2)^2}\,(\zeta-h_1) \tag{18.22a}$$

$$\tau^{*\alpha 3} = \tau^{*3\alpha} = \left(\frac{3\tau^{\alpha 3}}{2}\right)\left(1-\left(\frac{\zeta-h_1}{(h_1+h_2)/2}\right)^2\right) = \frac{3}{2}\,\tau^{\alpha 3}\left(1-\left(\frac{2(\zeta-h_1)}{(h_1+h_2)}\right)^2\right) \tag{18.22b}$$

$$\tau^{33} = 0 \tag{18.22c}$$

Introducing $(18.22)_{a,b,c}$ into (18.20) we proceed to obtain each term on the right-hand side of (18.20) as follows:

$$\int_o^{h_1} \frac{\nu_1^*}{E_1^*}\,\tau^{*\alpha\beta}\tau^{*\gamma\delta}d\zeta = \frac{\nu_1^*}{E_1^*}\int_o^{h_1} \left\{\left(\frac{12}{(h_1+h_2)^2}\right)^2 s^{\alpha\beta}s_{\gamma\delta}(\zeta-h_1)^2\right\}d\zeta$$

$$= \frac{\nu_1^*}{E_1^*}\left(\frac{12}{(h_1+h_2)^2}\right)^2 s^{\alpha\beta}s^{\gamma\delta}\int_o^{h_1}(\zeta^2-2h_1\zeta+h_1^2)d\zeta$$

$$= \frac{\nu_1^*}{E_1^*}\left(\frac{12}{(h_1+h_2)^2}\right)^2 s^{\alpha\beta}s^{\gamma\delta}\left[\frac{1}{3}\,\zeta^3-h_1\zeta^2+h_1^2\zeta\right]_o^{h_1}$$

$$= \frac{\nu_1^*}{E_1^*}\left(\frac{12}{(h_1+h_2)^2}\right)^2 s^{\alpha\beta}s^{\gamma\delta}\left(\frac{1}{3}\,h_1^3-h_1^3+h_1^3\right)$$

$$= 48\,\frac{\nu_1^*}{E_1^*}\,\frac{h_1^3}{(h_1+h_2)^4}\,s^{\alpha\beta}s^{\gamma\delta} \tag{18.23}$$

Moreover,

$$\int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{E_2^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta = \frac{\nu_2^*}{E_2^*} \int_{h_1}^{h_1+h_2} \{(\frac{12}{(h_1+h_2)^2})^2 s^{\alpha\beta}s^{\gamma\delta}(\zeta-h_1)^2\} d\zeta$$

$$= \frac{\nu_2^*}{E_2^*} (\frac{12}{(h_1+h_2)^2})^2 s^{\alpha\beta}s^{\gamma\delta} \int_{h_1}^{h_1+h_2} (\zeta-h_1)^2 d\zeta$$

$$= \frac{\nu_2^*}{E_2^*} \frac{(12)^2}{(h_1+h_2)^4} s^{\alpha\beta}s^{\gamma\delta}[\frac{1}{3} (\zeta-h_1)^3]_{h_1}^{h_1+h_2}$$

$$= \frac{\nu_2^*}{E_2^*} \frac{(12)^2}{(h_1+h_2)^4} s^{\alpha\beta}s^{\gamma\delta}[\frac{1}{3} h_2^3]$$

$$= 48 \frac{\nu_2^*}{E_2^*} \frac{h_2^3}{(h_1+h_2)^4} s^{\alpha\beta}s^{\gamma\delta} \tag{18.24}$$

Adding both sides of (18.23) and (18.24) we obtain

$$\int_{o}^{h_1+h_2} \frac{\nu^*}{E^*} \tau^{*\alpha\beta}\tau^{\gamma\delta} d\zeta = \int_{o}^{h_1} \frac{\nu_1^*}{E_1^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta + \int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{E_2^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta$$

$$= \frac{48}{(h_1+h_2)^4} (\frac{\nu_1^*}{E_1^*} h_1^3 + \frac{\nu_2^*}{E_2^*} h_2^3) s^{\alpha\beta}s^{\gamma\delta}$$

$$= \frac{48}{h} (\frac{\nu_1^*}{E_1^*} n_1^3 + \frac{\nu_2^*}{E_2^*} n_2^3) s^{\alpha}\beta s^{\gamma}\delta \tag{18.25}$$

where we have introduced a nondimensional parameter n such that

$$h = h_1 + h_2 \quad , \quad n_\alpha = \frac{h_\alpha}{h} \quad (\alpha = 1,2) \quad , \quad n = \frac{h_2}{h_1} \tag{18.26}$$

In a similar manner we may write

$$\int_{o}^{h_1+h_2} \frac{1+\nu^*}{E^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta = \int_{o}^{h_1} \frac{1+\nu_1^*}{E_1^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta + \int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{E_2^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta$$

$$= \frac{48}{(h_1+h_2)^4} \left(\frac{1+\nu_1^*}{E_1^*} h_1^3 + \frac{1+\nu_2^*}{E_2^*} h_2^3\right) s^{\alpha\beta} s^{\gamma\delta}$$

$$= \frac{48}{h} \left(\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3\right) s^{\alpha\beta} s^{\gamma\delta} \tag{18.27}$$

Next consider

$$\int_o^{h_1} \frac{1+\nu_1^*}{E_1^*} \tau^{*\alpha 3} \tau^{*\beta 3} d\zeta = \int_o^{h_1} \frac{1+\nu_1^*}{E_1^*} \{(\frac{3}{2})^2 \tau^{\alpha 3} \tau^{\beta 3} [1 - (\frac{2(\zeta-h_1)}{(h_1+h_2)})^2]^2\} d\zeta$$

$$= \frac{9}{4} \frac{1+\nu_1^*}{E_1^*} \tau^{\alpha 3} \tau^{\beta 3} \int_o^{h_1} \{1 - (\frac{2(\zeta-h_1)}{(h_1+h_2)})^2\}^2 d\zeta$$

$$= \frac{9}{4} \frac{1+\nu_1^*}{E_1^*} \tau^{\alpha 3} \tau^{\beta 3} \int_o^{h_1} \{1 - 2(\frac{2(\zeta-h_1)}{h_1+h_2})^2 + (\frac{2(\zeta-h_1)}{h_1+h_2})^4\} d\zeta$$

$$= \frac{9}{4} \frac{1+\nu_1^*}{E_1^*} \tau^{\alpha 3} \tau^{\beta 3} \int_o^{h_1} \{1 - 8 \frac{(\zeta-h_1)^2}{(h_1+h_2)^2} + 16 \frac{(\zeta-h_1)}{(h_1+h_2)^4}\} d\zeta$$

$$= \frac{9}{4} \frac{1+\nu_1^*}{E_1^*} \tau^{\alpha 3} \tau^{\beta 3} \{\zeta - \frac{8}{3} \frac{(\zeta-h_1)^3}{(h_1+h_2)^2} + \frac{16}{5} \frac{(\zeta-h_1)^5}{(h_1+h_2)^4}\}_o^{h_1}$$

$$= \frac{9}{4} \frac{1+\nu_1^*}{E_1^*} \tau^{\alpha 3} \tau_{\beta 3} \{h_1 - \frac{8}{3} \frac{h_1^3}{(h_1+h_2)^2} + \frac{16}{5} \frac{h_1^5}{(h_1+h_2)^4}\}$$

$$= \frac{9}{4} \frac{1+\nu_1^*}{E_1^*} h_1 \tau^{\alpha 3} \tau^{\beta 3} \{1 - \frac{8}{3} (\frac{h_1}{h_1+h_2})^2 + \frac{16}{5} (\frac{h_1}{h_1+h_2})^4\} \tag{18.28}$$

Moreover

$$\int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{E_2^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta = \int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{E_2^*} \{(\frac{3}{2})^2 \tau^{\alpha 3}\tau^{\beta 3}[1-(\frac{2(\zeta-\zeta_1)}{(h_1+h_2)})^2]^2\} d\zeta$$

$$= \frac{9}{4} \frac{1+\nu_2^*}{E_2^*} \tau^{\alpha 3}\tau^{\beta 3} \int_{h_1}^{h_1+h_2} [1-(\frac{2(\zeta-h_1)}{h_1+h_2})^2]^2 d\zeta$$

$$= \frac{9}{4} \frac{1+\nu_2^*}{E_2^*} \tau^{\alpha 3}\tau^{\beta 3} \int_{h_1}^{h_1+h_2} \{1-2(\frac{2(\zeta-h_1)}{h_1+h_2})^2+(\frac{2(\zeta-h_1)}{h_1+h_2})^4\} d\zeta$$

$$= \frac{9}{4} \frac{1+\nu_2^*}{E_2^*} \tau^{\alpha 3}\tau^{\beta 3} \{\zeta - \frac{8}{3} \frac{(\zeta-h_1)^3}{(h_1+h_2)^2} + \frac{16}{5} \frac{(\zeta-h_1)^5}{(h_1+h_2)^4}\}_{h_1}^{h_1+h_2}$$

$$= \frac{9}{4} \frac{1+\nu_2^*}{E_2^*} \tau^{\alpha 3}\tau^{\beta 3} \{(h_1+h_2) - \frac{8}{3} \frac{h_3^2}{(h_1+h_2)^2} + \frac{16}{5} \frac{h_2^5}{(h_1+h_2)^4} - h_1\}$$

$$= \frac{9}{4} \frac{1+\nu_2^*}{E_2^*} \tau^{\alpha 3}\tau^{\beta 3} \{h_2 - \frac{8}{3} \frac{h_2^3}{(h_1+h_2)^2} + \frac{16}{5} \frac{h_2^5}{(h_1+h_2)^4}\}$$

$$= \frac{9}{4} \frac{1+\nu_2^*}{E_2^*} h_2 \tau^{\alpha 3}\tau^{\beta 3} \{1 - \frac{8}{3} (\frac{h_2}{h_1+h_2})^2 + \frac{16}{5} (\frac{h_2}{h_1+h_2})^4\} \qquad (18.29)$$

From (18.28) and (18.29) it follows

$$\int_0^{h_1+h_2} \frac{1+\nu^*}{E^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta = \int_0^{h_1} \frac{1+\nu_1^*}{E_1^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta + \int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{E_2^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta$$

$$= \frac{9}{4} \frac{1+\nu_1^*}{E_1^*} h_1 \tau^{\alpha 3}\tau^{\beta 3} \{1 - \frac{8}{3} (\frac{h_1}{h_1+h_2})^2 + \frac{16}{5} (\frac{h_1}{h_1+h_2})^4\}$$

$$+ \frac{9}{4} \frac{1+\nu_2^*}{E_2^*} h_2 \tau^{\alpha 3}\tau^{\beta 3} \{1 - \frac{8}{3} (\frac{h_2}{h_1+h_2})^2 + \frac{16}{5} (\frac{h_2}{h_1+h_2})^4\}$$

$$= \frac{9}{4} \tau^{\alpha 3}\tau^{\beta 3} \{(\frac{1+\nu_1^*}{E_1{}^*} h_1 + \frac{1+\nu_2^*}{E_2^*} h_2) - \frac{8}{3} (\frac{1+\nu_1^*}{E_1^*} \frac{h_1^3}{(h_1+h_2)^2} + \frac{1+\nu_2^*}{E_2^*} \frac{h_2^3}{(h_1+h_2)^2})$$

$$+ \frac{16}{5} (\frac{1+\nu_1^*}{E_1^*} \frac{h_1^5}{(h_1+h_2)^4} + \frac{1+\nu_2^*}{E_2^*} \frac{h_2^5}{(h_1+h_2)^4})\}$$

$$= \frac{9}{4} h_1 \{(\frac{1+\nu_1^*}{E_1^*} + \frac{1+\nu_2^*}{E_2^*} n) - \frac{8}{3} (\frac{1+\nu_1^*}{E_1^*} \frac{1}{(1+n)^2} + \frac{1+\nu_2^*}{E_2^*} \frac{n^3}{(1+n)^2})$$

$$+ \frac{16}{5} (\frac{1+\nu_1^*}{E_1^*} \frac{1}{(1+n)^4} + \frac{1+\nu_2^*}{E_2^*} \frac{n^5}{(1+n)^4})\}\tau^{\alpha 3}\tau^{\beta 3}$$

$$= \frac{9}{4} h \{(\frac{1+\nu_1^*}{E_1^*} n_1 + \frac{1+\nu_2^*}{E_2^*} n_2) - \frac{8}{3} (\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3)$$

$$+ \frac{16}{5} (\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_2^*} n_2^5)\}\tau^{\alpha 3}\tau^{\beta 3} \qquad (18.30)$$

or

$$\int_0^{h_1+h_2} \frac{1+\nu^*}{E^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta = \frac{9h}{4} \{(n_1 - \frac{8}{3} n_1^3 + \frac{16}{5} n_1^5) \frac{1+\nu_1^*}{E_1^*}$$

$$+ (n_2 - \frac{8}{3} n_2^3 + \frac{16}{5} n_2^5) \frac{1+\nu_2^*}{E_2^*}\}\tau^{\alpha 3}\tau^{\beta 3} \qquad (18.31)$$

In view of (18.22c), all integrals in (18.20) involving τ^{*33} vanish identically. Substituting (18.25), (18.26) and (18.31) into (18.20) we obtain

$$\rho_o(h_1+h_2)\phi = q_1 A_{\alpha\beta}A_{\gamma\delta}\{\ \frac{48}{2h}\ (\frac{\nu_1^*}{E_1^*}\ n_1^3 + \frac{\nu_2^*}{E_2^*}\ n_2^3)s^{\alpha\beta}s^{\gamma\delta}\}$$

$$-\ q_2 A_{\alpha\gamma}A_{\beta\delta}\{\ \frac{48}{2h}\ (\frac{1+\nu_1^*}{E_1^*}\ n_1^3 + \frac{1+\nu_2^*}{E_2^*}\ n_2^3)s^{\alpha\beta}s^{\gamma\delta}\}$$

$$-\ q_4\ A_{\alpha\beta}\{\ \frac{9h}{4}\ [(n_1 - \frac{8}{3}\ n_1^3 + \frac{16}{5}\ n_1^5)\ \frac{1+\nu_1^*}{E_1^*} + (n_2 - \frac{8}{3}\ n_2^3 + \frac{16}{5}\ n_2^5)\ \frac{1+\nu_2^*}{E_2^*}]\}\tau^{\alpha 3}\tau^{\beta 3}$$

or

$$\rho_o(h_1+h_2)\phi = q_1\ \frac{24}{h}\ (\frac{\nu_1^*}{E_1^*}\ n_1^3 + \frac{\nu_2^*}{E_2^*}\ \nu_2^3)A_{\alpha\beta}A_{\gamma\delta}s^{\alpha\beta}s^{\gamma\delta}$$

$$-\ q_2\ \frac{24}{h}\ (\frac{1+\nu_1^*}{E_1^*}\ n_1^3 + \frac{1+\nu_2^*}{E_2^*}\ n_2^3)A_{\alpha\gamma}A_{\beta\delta}s^{\alpha\beta}s^{\gamma\delta}$$

$$-\ q_4\ \frac{9h}{4}\ [(n_1 - \frac{8}{3}\ n_1^3 + \frac{16}{5}\ n_1^5)\ \frac{1+\nu_1^*}{E_1^*}$$

$$+\ (n_2 - \frac{8}{3}\ n_2^3 + \frac{16}{5}\ n_2^5)\ \frac{1+\nu_2^*}{E_2^*}]A_{\alpha\beta}\tau^{\alpha 3}\tau^{\beta 3} \qquad (18.33)$$

Expression (18.32) is the Gibbs free energy for a composite laminate consisting of two elastic materials only.

For simplicity we introduce

BASE

$$C_1 = q_1 \frac{24}{h^2} \left(\frac{\nu_1^*}{E_1^*} n_1^3 + \frac{\nu_2^*}{E_2^*} n_2^3\right)$$

$$C_2 = q_2 \frac{24}{h^2} \left(\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3\right)$$

$$C_3 = 0 \tag{18.34}$$

$$C_4 = q_4 \frac{9}{4} \left[\left(\frac{1+\nu_1^*}{E_1^*} n_1 + \frac{1+\nu_2^*}{E_2^*} n_2\right) - \frac{8}{3} \left(\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3\right) + \frac{16}{5} \left(\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_2^*} n_2^5\right)\right]$$

$$C_5 = 0$$

Making use of (18 33) we may write (18.32) as follows

$$\rho_o\phi = C_1 A_{\alpha\beta} A_{\gamma\delta} s^{\alpha\beta} s^{\gamma\delta} - C_2 A_{\alpha\gamma} A_{\beta\delta} s^{\alpha\beta} s^{\gamma\delta} - C_4 A_{\alpha\beta} \tau^{\alpha 3} \tau^{\beta 3}$$

or

$$\rho_o\phi = (C_1 A_{\alpha\beta} A_{\gamma\delta} - C_2 A_{\alpha\gamma} A_{\beta\delta}) s^{\alpha\beta} s^{\gamma\delta} - C_4 A_{\alpha\beta} \tau^{\alpha 3} \tau^{\beta 3} \tag{18.35}$$

19. Constitutive coefficients for an initially flat composite laminate in extension

Considering the assumptions and development in section 18, we proceed to obtain the relevant constitutive coefficients for an initially flat composite laminate in extension. Guided by the theory of Cosserat surface, we introduce the following expressions for stresses

$$\tau^{*\alpha\beta} = \tau^{\alpha\beta} \tag{19.1.a}$$

$$\tau^{*\alpha 3} = \tau^{*3\alpha} = \frac{15}{(h_1+h_2)}\, s^{\alpha 3}\, \{\frac{2(\zeta-h_1)}{(h_1+h_2)} - (\frac{2(\zeta-h_1)}{(h_1+h_2)})^3\} \tag{19.1.b}$$

$$\tau^{*33} = k^3 \tag{19.1.c}$$

Introducing $(19.1)_{a,b,c}$ into (18.20) we proceed to obtain each term on the right-hand side of (18.20) as follows:

$$\int_o^{h_1} \frac{\nu_1^*}{2E_1^*}\, \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta = \int_o^{h_1} \frac{\nu_1^*}{2E_1^*}\, \{(\tau^{\alpha\beta})(\tau^{\gamma\delta})\} d\zeta$$

$$= \frac{\nu_1^*}{2E_1^*}\, \tau^{\alpha\beta}\tau^{\gamma\delta} \int_o^{h_1} d\zeta$$

$$= \frac{\nu_1^*}{2E_1^*}\, \tau^{\alpha\beta}\tau^{\gamma\delta}[\zeta]_o^{h_1} = \frac{\nu_1^*}{2E_1^*}\, h_1\, \tau^{\alpha\beta}\tau^{\gamma\delta} \tag{19.2}$$

and

$$\int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{2E_2^*}\, \tau^{*\alpha\beta}\tau^{*\gamma\delta}\, d\zeta = \int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{2E_2^*}\, \{(\tau^{\alpha\beta})(\tau^{\gamma\delta})\} d\zeta$$

$$= \frac{\nu_2^*}{2E_2^*}\, \tau^{\alpha\beta}\tau^{\gamma\delta}[\zeta]_{h_1}^{h_2+h_1}$$

$$= \frac{\nu_2^*}{2E_2^*}\, h_2\, \tau_{\alpha\beta}\tau^{\gamma\delta} \tag{19.3}$$

From (19.2) and (19.3) we have

$$\int_{o}^{h_1+h_2} \frac{\nu^*}{2E^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta = \frac{\nu_1^*}{2E_1^*} h_1 \tau^{\alpha\beta}\tau^{\gamma\delta} + \frac{\nu_2^*}{2E_2^*} h_2 \tau^{\alpha\beta}\tau^{\gamma\delta}$$

$$= \frac{1}{2} \left(\frac{\nu_1^*}{E_1^*} h_1 + \frac{\nu_2^*}{E_2^*} h_2\right)\tau^{\alpha\beta}\tau^{\gamma\delta}$$

$$= \frac{h}{2} \left(\frac{\nu_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2\right)\tau^{\alpha\beta}\tau^{\gamma\delta} \tag{19.4}$$

Similarly we obtain

$$\int_{o}^{h_1} \frac{1+\nu_1^*}{2E_1^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta = \int_{o}^{h_1} \frac{1+\nu_1^*}{2E_1^*} \{(\tau^{\alpha\beta})(\tau^{\gamma\delta})\} d\zeta$$

$$= \frac{1+\nu_1^*}{2E_1^*} \tau^{\alpha\beta}\tau^{\gamma\delta} \int_{o}^{h_1} d\zeta$$

$$= \frac{1+\nu_1^*}{2E_1^*} \tau^{\alpha\beta}\tau^{\gamma\delta} [\zeta]_{o}^{h_1} = \frac{1+\nu_1^*}{2E_1^*} h_1 \tau^{\alpha\beta}\tau^{\gamma\delta} \tag{19.5}$$

and

$$\int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{2E_2^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta = \int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{2E_2^*} \{(\tau^{\alpha\beta})(\tau^{\gamma\delta})\} d\zeta$$

$$= \frac{1+\nu_2^*}{2E_2^*} \tau^{\alpha\beta}\tau^{\gamma\delta} [\zeta]_{h_1}^{h_1+h_2}$$

$$= \frac{1+\nu_2^*}{2E_2^*} h_2 \tag{19.6}$$

From (19.5) and (19.6) we have

$$\int_{o}^{h_1+h_2} \frac{1+\nu^*}{2E^*} \tau^{*\alpha\beta}\tau^{*\gamma\delta} d\zeta = \frac{1+\nu_1^*}{2E_1^*} h_1 \tau^{\alpha\beta}\tau^{\gamma\delta} + \frac{1+\nu_2^*}{2E_2^*} h_2 \tau^{\alpha\beta}\tau^{\gamma\delta}$$

$$= \frac{h}{2}\left(\frac{1+\nu_1^*}{E_1^*} n_1 + \frac{1+\nu_2^*}{E_2^*} n_2\right)\tau^{\alpha\beta}\tau^{\gamma\delta} \tag{19.7}$$

Moreover,

$$\int_o^{h_1} \frac{1}{2E_1^*} (\tau^{*33})^2 d\zeta = \int_o^{h_1} \frac{1}{2E_1^*} (k^3)^2 d\zeta = \frac{1}{2E_1^*} (k^3)^2 \int_o^{h_1} d\zeta$$

$$= \frac{1}{2E_1^*} (k^3)^2 [\zeta]_o^{h_1} = \frac{1}{2E_1^*} h_1 (k^3)^2 \tag{19.8}$$

and

$$\int_{h_1}^{h_1+h_2} \frac{1}{2E_2^*} (\tau^{*33})^2 d\zeta = \int_{h_1}^{h_1+h_2} \frac{1}{2E_2^*} \left(\frac{k^3}{h_1+h_2}\right)^2 d\zeta = \frac{1}{2E_2^*} (k^3$$

$$= \frac{1}{2E_2^*} h_2 (k^3)^2 \tag{19.9}$$

From (19.8) and (19.9) we obtain

$$\int_o^{h_1+h_2} \frac{1}{2E^*} (\tau^{*33})^2 d\zeta = \frac{1}{2E_1^*} h_1 (k^3)^2 + \frac{1}{2E_2^*} h_2 (k^3)^2$$

$$= \frac{h}{2}\left(\frac{1}{E_1^*} n_1 + \frac{1}{E_2^*} n_2\right)(k^3)^2 \tag{19.10}$$

In addition we have

$$\int_o^{h_1} \frac{\nu_1^*}{E_1^*} \tau^{*\alpha\beta}\tau^{*33} d\zeta = \int_o^{h_1} \frac{\nu_1^*}{E_1^*} (\tau^{\alpha\beta})(k_3) d\zeta$$

$$= \frac{\nu_1^*}{E_1^*} \tau^{\alpha\beta} k^3 \int_o^{h_1} d\zeta = \frac{\nu_1^*}{E_1^*} \tau^{\alpha\beta} k^3 [\zeta]_o^{h_1}$$

$$= \frac{\nu_1^*}{E_1^*} h_1 \tau^{\alpha\beta} k^3 \tag{19.11}$$

and

$$\int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{E_2^*} \tau^{*\alpha\beta}\tau^{*33}d\zeta = \int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{E_2^*} (\tau^{\alpha\beta})(k^3)d\zeta$$

$$= \frac{\nu_2^*}{E_2^*} \tau^{\alpha\beta}k^3 [\zeta]_{h_1}^{h_1+h_2} = \frac{\nu_2^*}{E_2^*} h_2 \tau_{\alpha\beta}k^3 \tag{19.12}$$

From (19.11) and (19.12) we obtain

$$\int_{o}^{h_1+h_2} \frac{\nu^*}{E^*} \tau^{*\alpha\beta}\tau^{*33}d\zeta = \frac{\nu_1^*}{E_1^*} h_1 \tau^{\alpha\beta}k^3 + \frac{\nu_2^*}{E_2^*} h_2 \tau_{\alpha\beta}k^3$$

$$= h\,(\frac{\nu_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2)\tau^{\alpha\beta}k^3 \tag{19.13}$$

Finally we write

$$\int_o^{h_1} \frac{1+\nu_1^*}{E_1^*} \tau^{*\alpha 3}\tau^{*\beta 3}d\zeta = \frac{1+\nu_1^*}{E_1^*} \int_o^{h_1} [\frac{15}{(h_1+h_2)}]^2 s^{\alpha 3}s^{\beta 3}\{\frac{2(\zeta-h_1)}{(h_1+h_2)} - (\frac{2(\zeta-h_1)}{(h_1+h_2)})^3]^2 d\zeta$$

$$= \frac{1+\nu_1^*}{E_2^*} \frac{(15)^2}{(h_1+h_2)^2} s^{\alpha 3}s^{\beta 3} \int_o^{h_1} \{\frac{4(\zeta-h_1)^2}{(h_1+h_2)^2} - \frac{32(\zeta-h_1)^4}{(h_1+h_2)^4} + \frac{64(\zeta-h_1)^6}{(h_1+h_2)^6}\}d\zeta$$

$$= \frac{1+\nu_1^*}{E_2^*} \frac{(30)^2}{(h_1+h_2)^4} s^{\alpha 3}s^{\beta 3} \int_o^{h_1} \{(\zeta-h_1)^2 - \frac{8(\zeta-h_1)^4}{(h_1+h_2)^1} + \frac{16(\zeta-\mathrm{hsub1})^6}{(h_1+h_2)^4}\}d\zeta$$

$$= \frac{1+\nu_1^*}{E_2^*} \frac{(30)^2}{(h_1+h_2)^4} s^{\alpha 3}s^{\beta 3}[\frac{(\zeta-h_1)^3}{3} - \frac{8}{5}\frac{(\zeta-h_1)^5}{(h_1-h_2)^2} + \frac{16}{7}\frac{(\zeta-h_1)^7}{(h_1+h_2)^4}]_o^{h_1}$$

$$= \frac{1+\nu_1^*}{E_2^*} \frac{(30)^2}{(h_1+h_2)^4} s^{\alpha 3}s^{\beta 3}[\frac{h_1^3}{3} - \frac{8}{5}\frac{h_1^5}{(h_1+h_2)^2} + \frac{16}{7}\frac{h_1^7}{(h_1+h_2)^4}]$$

$$= \frac{1+\nu_1^*}{E_2^*} \frac{(30)^2}{h} s^{\alpha 3}s^{\beta 3}[\frac{1}{3}(\frac{h_1}{h_1+h_2})^3 - \frac{8}{5}(\frac{h_1}{h_1+h_2})^5 + \frac{16}{7}(\frac{h_1}{h_1+h_2})^7]$$

$$= \frac{900}{h} \frac{1+\nu_1^*}{E_1^*} (\frac{1}{3} n_1^3 - \frac{8}{5} n_1^5 + \frac{16}{7} n_1^7)s^{\alpha 3}s^{\beta 3} \tag{19.14}$$

In a similar manner we obtain

$$\int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{E_2^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta = \frac{900}{h} \frac{1+\nu_2^*}{E_2^*} \left(\frac{1}{3} n_2^3 - \frac{8}{5} n_2^5 + \frac{16}{7} n_2^7\right) s^{\alpha 3} s^{\beta 3} \qquad (19.15)$$

By (19.14) and (19.15) we have

$$\int_{h_1}^{h_1+h_2} \frac{1+\nu^*}{E^*} \tau^{*\alpha 3}\tau^{*\beta 3} d\zeta = \frac{900}{h} \frac{1+\nu_1^*}{E_1^*} \left(\frac{1}{3} n_1^3 - \frac{8}{5} n_1^5 + \frac{16}{7} n_1^5\right) s^{\alpha 3} s^{\beta 3}$$

$$+ \frac{900}{h} \frac{1+\nu_2^*}{E_2^*} \left(\frac{1}{3} n_2^3 - \frac{8}{5} n_2^5 + \frac{16}{7} n_2^7\right) s^{\alpha 3} s^{\beta 3}$$

$$= \frac{900}{h} \left\{ \frac{1}{3} \left(\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3\right) - \frac{8}{5} \left(\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_2^*} n_2^5\right)\right.$$

$$\left. + \frac{16}{7} \left(\frac{1+\nu_1^*}{E_1^*} n_1^7 + \frac{1+\nu_2^*}{E_2^*} n_2^7\right)\right\} s^{\alpha 3} s^{\beta 3} \qquad (19.16)$$

Substituting (19.4), (19.7), (19.10), (19.13) and (19.16) into (18.20), we obtain

$$\rho_o(h_1+h_2)\phi = q_1 A_{\alpha\beta} A_{\gamma\delta} \left\{ \frac{h}{2} \left(\frac{\nu_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2\right)\right\} \tau^{\alpha\beta}\tau^{\gamma\delta}$$

$$- q_2 A_{\alpha\gamma} A_{\beta\delta} \left\{ \frac{h}{2} \left(\frac{1+\nu_1^*}{E_1^*} n_1 + \frac{1+\nu_2^*}{E_2^*} n_2\right)\right\} \tau^{\alpha\beta}\tau^{\gamma\delta}$$

$$+ q_3 A_{\alpha\beta} \left\{ h \left(\frac{\nu_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2\right)\right\} \tau^{\alpha\beta} k^3$$

$$- q_4 A_{\alpha\beta} \left\{ \frac{900}{h} \left[\frac{1}{3} \left(\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3\right) - \frac{8}{5} \left(\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_2^*} n_2^5\right)\right.\right.$$

$$\left.\left. + \frac{16}{7} \left(\frac{1+\nu_1^*}{E_1^*} n_1^7 + \frac{1+\nu_2^*}{E_2^*} n_2^7\right)\right]\right\} s^{\alpha 3} s^{\beta 3}$$

$$- q_5 \left\{ \frac{h}{2} \left(\frac{1}{E^*} n_1 + \frac{1}{E^*} n_2\right)\right\} (k^3)^2$$

or

BASE

$$\rho_o(h_1+h_2)\phi = q_1 \frac{h}{2} \frac{\nu_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2) A_{\alpha\beta} A_{\gamma\delta} \tau^{\alpha\beta} \tau^{\gamma\delta}$$

$$- q_2 \frac{h}{2} (\frac{1+\nu_1^*}{E_1^*} n_1 + \frac{1+\nu_2^*}{E_2^*} n_2) A_{\alpha\gamma} A_{\beta\delta} \tau^{\alpha\beta} \tau^{\gamma\delta}$$

$$+ q_3 h (\frac{n_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2) A_{\alpha\beta} \tau^{\alpha\beta} k^3$$

$$- q_4 \frac{900}{h} [\frac{1}{3} (\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3) - \frac{8}{5} (\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1-\nu_2^*}{E_2^*} n_2^5)$$

$$+ \frac{16}{7} (\frac{1+\nu_1^*}{E_1^*} n_1^7 + \frac{1+\nu_2^*}{E_2^*} n_2^7)] A_{\alpha\beta} s^{\alpha 3} s^{\beta 3}$$

$$- q_5 \frac{h}{2} (\frac{1}{E_1^*} n_1 + \frac{1}{E_2^*} n_2)(k^3)^2 \tag{19.17}$$

For simplicity we introduce

$$D_1 = q_1 \frac{1}{2} (\frac{\nu_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2)$$

$$D_2 = q_2 \frac{1}{2} (\frac{1+\nu_1^*}{E_1^*} n_1 + \frac{1+\nu_2^*}{E_2^*} n_2)$$

$$D_3 = q_3 \, 1 \, (\frac{\nu_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2)$$

$$D_4 = q_4 \frac{900}{h^2} [\frac{1}{3} (\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3) - \frac{8}{5} (\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_2^*} n_2^5)$$

$$+ \frac{16}{7} (\frac{1+\nu_1^*}{E_1^*} n_1^7 + \frac{1+\nu_2^*}{E_2^*} n_2^7)]$$

$$D_5 = q_5 \frac{1}{2} (\frac{1}{E_1^*} n_1 + \frac{1}{E_2^*} n_2)$$

With the help of (19.18) we may rewrite (19.17) as follows:

$$\rho_o\phi = D_1 A_{\alpha\beta} A_{\gamma\delta} \tau^{\alpha\beta} \tau^{\gamma\delta} - D_3 A_{\alpha\gamma} A_{\beta\delta} \tau^{\alpha\beta} \tau^{\gamma\delta} + D_3 A_{\alpha\beta} \tau^{\alpha\beta} k^3 - D_4 A_{\alpha\beta} s^{\alpha 3} s^{\beta 3} - D_5 (k^3)^2$$

or

$$\rho_o\phi = (D_1 A_{\alpha\beta} A_{\gamma\delta} - D_2 A_{\alpha\gamma} A_{\beta\delta}) \tau^{\alpha\beta} \tau^{\gamma\delta} + D_3 A_{\alpha\beta} \tau^{\alpha\beta} k^3 - D_4 A_{\alpha\beta} s^{\alpha 3} s^{\beta 3} - D_5 (k^3)^2 \quad (19.19)$$

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BASE

A General Theory for Composite Laminates

by

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Introduction

Composite materials have fully established themselves as workable engineering materials. Early military application during World War II led to large-scale commercial and aerospace utilization. Today, industries such as aircraft, automobiles, sporting goods, electronics, and appliances are quite dependent on composite materials. Since advance composite materials offer significant advantages in terms of efficiency and cost. A widespread and efficient application of composite materials requires detailed and reliable knowledge of their physical properties and, in turn, of their behavior under the influence of external effects such as forces, thermal changes, etc. Because of potentially diverse structural and physical variety of reinforced composites, it is neither practical nor economical to rely solely on experimental determination of their properties. Therefore, similar to any other branch of physical sciences, it is desirable to develop a theory (or theories) so that we can analyze, explain, and predict the behavior of composite materials under various in-use loading conditions.

Generally speaking, composite materials are based on the concept of compounding reinforcing elements and matrix materials such that they form a reinforced composite. The mechanical behavior of such materials is termed mechanics of composite materials. More specifically, a composite material is one in which two or more constituents are combined to produce a new material with mechanical properties different from those of the individual constituents. It is assumed that the constituents of a composite material retain their individual chemical and mechanical integrity and characteristics. A typical composite material consists of a bounding, or matrix material containing a second reinforcing material in the form of continuous or discontinuous filaments or laminations. Major parameters involved in mechanics of composite materials are: volume fractions of reinforcing elements and matrix, direction of reinforcement, geometry of reinforcing elements and position of reinforcing elements relative to each other. Additional variety stems from the physical properties of the constituents. Altogether, the variation of the geometrical and physical parameters can lead to an enormous number of possibilities. It is,

therefore, desirable to have theories that can describe the physical behavior of the composite in terms of the known geometrical layout of the composite and the known physical properties of the consituents.

An appropriate classification of the mechanics of composite materials may be brought about by the definition of two areas of composite material behavior as follows:

a) Macromechanics: The study of composite material behavior wherein the material is presumed homogeneous and the effect of the constituent materials are detected only as averaged apparent properties of the composite.

b) Micromechanics: The study of composite material behavior wherein the interaction of the constituent materials is examined in detail as part of the behavior of the heterogeneous composite material.

The properties of a lamina can be experimentally determined in the "as made" state or can be mathematically derived on the basis of the properties of the constituent materials. That is, we can predict lamina properties by the procedures of micromechanics and we can measure lamina properties by physical means and use the properties in a micromechanical analysis of the structure. Knowledge of how to predict properties is essential in constructing composites that have certain apparent or macroscopic properties. Thus, micromechanics is a natural compliment to macromechanics, and the formulation of an adequate (continuum) theory that could describe the mechanical behavior considering the micro-structure of composite materials is highly desirable and of major concern in material engineering due to many advantages that composite materials offer in terms of cost, weight and peformance.

In the last three decades several continuum theories have been proposed as models of elastostatics or elastodynamics of composite materials. In general these theories may be divided into two major categories as follows:

1) theories that do not account for the effect of microstructure.

2) theories that consider the behavior of microstructure and try to account for its effect in continuum.

The so-called "effective modulus" theories replace the actual composite by a homogeneous, generally anisotropic medium whose material constants are a geometrically weighted average of the properties of the constituents. While yielding satisfactory results for certain geometries under static loads, such an approach exhibits serious deficiencies for virtually all geometries when applied to dynamic problems such as impact and wave propagation. Specifically, effective modules theories are incapable of reproducing the dispersion and attenuation observed in composite materials. Such a behavior is a well known phenomena in composites and is a result of the microstructure of the particular composite. The dynamic behavior of a composite material is of great importance when the material is subjected to high-rate loads such as the ones that are generated by impact or explosive charges. We briefly elaborate on this point.

The dynamic response of deformable heterogeneous materials may be broadly classified into two groups as follows.

i) The wave length of the characteristic response of the material is very long compared with the scale of the inhomogeneity. Then the material response is governed by the effective properties of the equivalent homogeneous medium. In this case the structural response and wave propagation are identical to those of homogeneous materials.

ii) The wave length of the response is not ideally long with respect to the characteristic dimension of the inhomogeneity. In this situation very complicated dynamic effects can occur. The interfaces between material phases cause wave reflection and refraction. This phenomena is due to the existence of microstructure in the composite.

Considering (i) and (ii) above, it is clear that any continuum theory designed to account for the dynamic response of a composite must, in some fashion, reflect the effect of microstructure in the composite. In addition to dynamic response of the composite laminates the issue of interlaminar behavior of composite laminates is of great importance. This issue is directly related to delamination and edge effects in composites. In recent years, delamination has become one of the most feared failure modes in laminated composite structures. It can exhibit unstable crack growth, and while delamination failure itself is not a catastrphic event, it can perpetrate such a condition due to its weakening influence on a component in its resistance to subsequent failure modes. This problem has also initiated a great deal of research in the field of composite laminates. What began in 1970 as somewhat of an academic curiosity turned into a beehive of research activity in recent years. This in turn indicates the desire for having an adequate theory that can account for the effect of interlaminar stresses.

With the rapid advancements in numerical analysis and computation power in recent years it would seen natural to take advantage of various numerical methods such as finite element method(s), etc. Indeed, there has been considerable effort in this direction. However, various specially developed composite shell and composite solid elements that are available today are not adequate for advanced applications. These elements are formulated using one or another form of shell theories and although they provide acceptable results in special situations and under simple loading conditions, they are not capable for predicting accurate structural response for extreme in use loading conditions. One reason for this deficiency has been the lack of a powerful theory for composite laminates. On the other hand, the use of the usual three dimensional solid elements is not a viable approach due to special geometry of composite laminates. The geometric characteristic of composite laminates requires an enormous number of elements to be used. This in turn requires an extremely high computational power and hence becomes quite expensive and time consuming, especially in the case of "thick" composite laminates. In general, our knowledge of the mechanical behavior of composite materials has lagged behind advances in such other related areas as increasing sophisticated computers and computational

methods. It is believed that for a reliable structural analysis, the model(s) of composite laminates should be based on sound thermomechanical principles.

A review of the literature on continuum theories developed for composite laminates reveals that most of the theories that, in some fashion, account for the effect of micro-structure are linear in nature. Consequently, these theories are not capable to model the behavior of composite laminates undergoing large deformation. Moreover, all continuum theories, with the exception of one [Blinowski, 1986], are proposed for composite laminates with initially flat configurations. Hence, these theories are not appropriate for curved geometries. In addition the available continuum theories are mainly developed to predict only the dynamic response of the composite laminates. Therefore, they do not seem to be adequate for problems involving static response of the laminated composites with specified boundary conditions, delamination and edge effects. Indeed the literature on the proposed continuum theories of composite laminates may be divided into two groups. One group is concerned with the formulation of theories that are adequate for dynamic response of composite laminates and another group that is involved with the formulation of theories that are appropriate for intelaminar response of composite materials.

The dynamic behavior of a composite material (or a continuum in general) is of great importance when the material is subjected to high-rate loads such as the ones generated by impact or explosive charges. The importance of the dynamic behavior of composite materials has directed the efforts of the researchers towards the consideration and incorporation of the effect of microstructure in their suggested continuum theories. One of the earliest efforts is due to Sun, Achenbach and Herrmann [Sun et al. 1968] in which they developed a linear continuum theory for a composite laminate. In this work and its subsequent refinements by Achenbach [1974, 1975], a system of displacement equations of motions was presented, pertaining to a continuum theory to describe the dynamic behavior of a laminated composite. In deriving the equations of motion the displacements of the reinforcing layers and the amtrix layers were expressed as two-term expansion about the mid-planes of the layers. Dynamic interaction of the layers was

included through continuity relations at the interfaces. By making use of the expression for the total energy of the composite and application of Hamilton's principle, where the continuity relations were included through the use of Lagrangian multipliers, the displacement equations of motion were obtained.

A generalization of the foregoing theory was given by Aboudi [1981a, 1983, 1985, 1989], who also expressed the displacement components in terms of Legendre polynomials but in somewhat more general terms. In his work Aboudi has imposed the condition of continuity of displacements and stress components between the adjacent layers where all the continuity conditions are satisfied pointwise throughout the common boundary of the adjacent layers. Aboudi has expanded on his theory and has been able to obtain a more generalized version applicable to nonelastic laminated exposites [1981b, 1985b].

Another approach to continuum formulation of laminated composites is due to Hegemier and Nayfeh. These authors developed a continuum theory for linear wave propagation normal to the layers of a laminated composite with elastic, periodic microstructure. Their construction is based upon an asymptotic scheme in which dominant signal wavelengths were assumed large compared to typical composite microdimensions.

A different approach to the problem was adopted by Bedford and Stern in which they presented a linear continuum theory that is based on the continuum theory of mixtures. The constituents of a composite were modeled as superimposed continua which undergo individual deformations. Effect of structure on dynamical processes in composite materials were then simulated by specified by the coupling between the individual constituent motions. The misture theory has also been used by other researchers to develop continuum theories for composite laminates. In this connection mention should be made of the work of Murakami and Hegemier [ ], Tiersten and McCarthy [ ] and McNiven [ ].

Yet another approach, namely variational formulations, was presented by Nemat-Nasser, Fu and Minagawa [ ], which was based on a quotient due to Nemat-Nasser. By making use of the quotient lower and upper bounds for the eigenfrequencies were then developed. This approach was later exploited and utilized with the help of finite element analysis, to study the harmonic waves of layered media.

Two other important and related issues regarding composite laminates are delamination and edge effects caused by interlaminar stresses in composite laminates. These issues have not been addressed either by the effective modulus theories, or by the various suggested dynamical continuum theories. Yet, due to the significance of interlaminar response of composite laminates a large amount of research is directed towards the development of theories that can treat the interlaminar response of composite laminates. In this connection, mention should be made of the pioneering work of Pagano and pipes.

Although some interesting and partial information has been produced in regard to delamination and edge effects by the available theories, these theories are all ad hoc, limited in their capabilities, and fail to account for dynamical effects.

All in all, it seems that the available theories are limited in their degrees of capability and hence lack the generality and applicability expected from a comprehensive continuum theory. The aforementioned issues become more important in the case of thick composite laminate, i.e., when the configuration of the composite approaches a three-dimensional body.

Considering the restrictions of existing theories, the specific objectives of this investigation has been to develop a general continuum theory for laminated composite materials that could account for the effect of i) microstructure, ii) nonlinearity, iii) curved geometry, and iv) interlaminar stresses. The theory presented here is represented by a set of well-defined conservation laws predicated on physical observations. Within the context of purely mechanical theory, the developed theory exhibits the following features:

It is well known that two different approaches may be adopted for the construction of mechanical theories appropriate for shell-like and rod-like bodies. One approach starts with the three-dimensional equations of classical continuum mechanics and by introducing special assumptions/approximations, attempts to obtaine one-dimensional or two-dimensional field equations and constitutive equations. In the second approach, the rod-like or shell-like body is represented (directly) by a purely one-dimensional or two-dimensional theory without any reference to the three-dimensional theory. In this approach, one proceeds to develop the field equations and the relevant constitutive equations directly.

The two foregoing approaches also apply to the construction of a theory for composite laminate due to the existence of special shell-like geometry. In this paper we adopt the first approach, namely derivation from the three-dimensional theory. The direct approach to the problem will be discussed in a separate paper. Moreover, in order to keep the paper in a manageable size, we confine our attention to the development of the nonlinear theory only. The derivation of the corresponding linear theory and other related issues are considered in another papeer.

2. Preliminaries. General background.

In this section we introduce the coordinate systems, and the notations which will be used in the subsequent development. We also record some relevant results from classical three-dimensional continuum mechanics.

2.1 Coordinate Systems

Let the points of a region $\mathcal{R}$ in a three dimensional Euclidean space be referred to a fixed right-handed rectangular Cartesian coordinate system x^i $(i = 1,2,3)$ and let η^i $(i = 1,2,3)$ be a general *convected* curvilinear coordinate system defined by the transformation

$$x^i = x^i(\eta^1, \eta^2, \eta^3) \tag{2.1}$$

We assume the above transformation is nonsingular in $\mathcal{R}$, i.e.,

$$\det\left(\frac{\partial x^i}{\partial \eta^i}\right) \neq 0 \tag{2.2}$$

This implies the existence of the unique inverse such that

$$\eta^i = \eta^i(x^1, x^2, x^3) \tag{2.3}$$

We recall that a convected coordinate system is normally defined in relation to a continuous body and moves continuously with the body throughout the motion of the body from one configuration to another.

Throughout this work, all Latin indices (subscripts or superscripts) take the values 1,2,3; all Greek indices (subscripts or superscripts) take the values 1,2 and the usual summation convention is employed. We will use a comma for partial differentiation with respect to either space or surface coordinates such as η^i or η^α and a superposed dot for material time derivative, i.e., differentiation with respect to time holding the material coordinates, such as η^i or η^α, fixed. Also,

we use a vertical bar (|) or a double vertical bar (||) for covariant differentiation in 2 and 3 dimensional spaces, respectively. Also, for convenience, often we set $\eta^3 = \xi$ and adopt the notation

$$\eta^i = (\eta^\alpha, \xi) \tag{2.4}$$

As it becomes clear shortly, in order to adequately represent the behavior of composite laminate we need to introduce a second system of convected coordinates. We will designate this latter system by θ^i ($i = 1,2,3$). In addition, in the course of derivation of various results for the composite laminate we will encounter covariant differentiation with respect to a coordinate system which corresponds to composite continuum. To denote this we will use a single boldfaced vertical bar (**|**). In what follows, when there is a possibility of confusion, quantities which represent the same physical/geometrical concepts will be denoted by the same symbol but with an added asterisk (*) for classical three dimensional continuum mechanics or an added hat (^) for the micro-structure and no addition for composite laminate (macro-structure). For example, the mass densities of a body in the contexts of the classical continuum mechanics, the Cosserat surface (micro-structure) and the composite laminate (macro-structure) will be denoted by ρ^*, $\hat{\rho}$ and ρ, respectively.

2.2. Basic equations of classical continuum mechanics in general curivlinear coordinates

In this section we summarize some preliminary results from the three-dimensional theory for non-polar media in terms of general curvilinear coordinates.

We define a body, designated by $\mathcal{B}^*$, as a set of particles (material points).[1] We designate the particles of the body by P^* and assume that the body is smooth and can be put into correspondence with a domain of the three-dimensional Euclidean space. Thus, by assumptions, a particle P^* of the body can be put into a one-to-one correspondence with the triples of real numbers P_1, P_2, P_3 in a region of Euclidean 3-space. We assume the mapping from the body manifold to the domain of a Euclidean 3-space is one-to-one, invertible, and differentiable as many times as desired.

Consider a body $\mathcal{B}^*$ with its particles P^* and its boundary $\partial\mathcal{B}^*$ (a closed surface) be embedded in a region $\mathcal{R}$ of the Euclidean 3-space, and let the particles (material points) of $\mathcal{B}^*$ be identified by a convected coordinate system (2.2). Let $\mathbf{P}^*$ denote the position vector, relative to a fixed origin, say 0, of a typical particle of $\mathcal{B}^*$ in a reference configuration. Then, we have[2]

$$\mathbf{P}^* = \mathbf{P}^*(\eta^\alpha, \xi) \tag{2.5}$$

This, in view of (2.2), may also be expressed as a function of x^i. We recall that, in general, the numerical values of the coordinates associated with each material point of a continuum varies from one configuration to another. However, when the particles of a continuum are referred to a convected coordinate system, the numerical values of the coordinates of a particle remain the same for all time. The position vector of a typical particle of $\mathcal{B}^*$ in the present (deformed) configuration at time t, relative to the same fixed origin will be denoted by

[1] Note that from now on when we refer to a body in the sense of classical continuum mechanics, we will denote it by an added asterisk (*). The same will be true for the various quantities associated with the body.

[2] Throughout this work for the sake of simplicity, and if there is no possibility of confusion, we denote a function and its value with the same symbol.

$$\mathbf{p}^* = \mathbf{p}^*(\eta^i,t) = \mathbf{p}^*(\eta^\alpha,\xi,t) \tag{2.6}$$

We note that equation (2.5) specifies the place occupied by the material point η^i in a reference configuration, while the place occupied by the same material point η^i in the present (deformed) configuration is specified by (2.6). We assume that the vector function $\mathbf{p}^*$ in (2.6), which describes the motion of the body $\mathcal{B}^*$ is differentiable with respect to η^α, ξ and t as many times as may be required. We recall the formulae

$$\begin{gathered} \mathbf{g}_i^* = \frac{\partial \mathbf{p}^*}{\partial \eta^i} \ , \ g_{ij}^* = \mathbf{g}_i^* \cdot \mathbf{g}_j^* \ , \ g^* = \det(g_{ij}^*) \ , \ g^{*1/2} = [\mathbf{g}_1^* \, \mathbf{g}_2^* \, \mathbf{g}_3^*] > 0 \ , \\ \mathbf{g}^{*i} = g^{*ij}\, \mathbf{g}_j^* \ , \ \mathbf{g}^{*i} \cdot \mathbf{g}^{*j} = g^{*ij} \ , \ \mathbf{g}^{*i} \cdot \mathbf{g}_j^* = \delta^i{}_j \end{gathered} \tag{2.7}$$

where $\mathbf{g}_i^*$ and $\mathbf{g}^{*i}$ are the covariant and the contravariant base vectors at time t, g_{ij}^* is the metric tensor, g^{*ij} is its conjugate, $\delta^i{}_j$ is the Kronecker symbol in 3-dimensional space, and [ ] denotes scalar triple product. We note that in any given motion $[\mathbf{g}_1^* \, \mathbf{g}_2^* \, \mathbf{g}_3^*]$ is either positive or negative. The choice of positive sign in $(2.7)_4$ is for definiteness and has the advantage that it requires η^i be a right-handed coordinate system. We also recall the expression for the square of a line element and an element of volume in the present configuration,

$$ds^2 = d\mathbf{p}^* \cdot d\mathbf{p}^* = g_{ij}^* d\eta^i d\eta^j \tag{2.8}$$

$$dv^* = g^{*1/2}\, d\eta^1\, d\eta^2\, d\eta^3 \tag{2.9}$$

Formulae analogous to (2.7) and (2.9), valid in a reference configuration are given by

$$\begin{gathered} \mathbf{G}_i^* = \frac{\partial \mathbf{P}^*}{\partial \eta^i} \ , \ G_{ij}^* = \mathbf{G}_i^* \cdot \mathbf{G}_j^* \ , \ G^* = \det(G_{ij}^*) \ , \ G^{*1/2} = [\mathbf{G}_1^* \, \mathbf{G}_2^* \, \mathbf{G}_3^*] > 0 \ , \\ \mathbf{G}^{*i} = G^{*ij}\, \mathbf{G}_j^* \ , \ \mathbf{G}^{*i} \cdot \mathbf{G}^{*j} = G^{*ij} \ , \ \mathbf{G}^{*i} \cdot \mathbf{G}_j^* = \delta^i{}_j \end{gathered} \tag{2.10}$$

and

$$dS^2 = d\mathbf{P}^* \cdot d\mathbf{P}^* = G_{ij}^*\, d\eta^i d\eta^i \tag{2.11}$$

$$dV^* = G^{*1/2}\, d\eta^1\, d\eta^2\, d\eta^3 \tag{2.12}$$

We define a strain measure through

$$ds^2 - dS^2 = 2\gamma^*_{ij}\, d\eta^i d\eta^j \tag{2.13}$$

$$\gamma^*_{ij} = \tfrac{1}{2}(g^*_{ij} - G^*_{ij}) \tag{2.14}$$

where G^*_{ij} is the metric tensor associated with the reference configuration and γ^*_{ij} are the covariant components of the symmetric strain tensor. Moreover, the velocity is given by

$$\mathbf{v}^* = \dot{\mathbf{p}}^* = \frac{\partial \mathbf{p}^*}{\partial t} \tag{2.15}$$

Let $\mathcal{P}^*$, bounded by a closed surface $\partial\mathcal{P}^*$, refer to an arbitrary part of the body $\mathcal{B}^*$ in the present configuration. Then within the scope of the classical (nonpolar) continuum mechanics, the system of forces acting over any part $\mathcal{P}^*$ of the body $\mathcal{B}^*$ in motion consists of the sum of the two types of forces, $\mathbf{F}^*_b$ and $\mathbf{F}^*_c$, as described below:

Let $\mathbf{b}^* = \mathbf{b}^*(\eta^i,t)$ be a vector field, per unit mass ρ^*, defined for material points in the region of the Euclidean space, occuped by $\mathcal{B}^*$ at time t. This vector field is called the *body force*. The *resultant body force* acting on the part $\mathcal{P}^*$ in the present configuration at time t is defined by

$$\mathbf{F}^*_b = \int_{\mathcal{P}^*} \rho^* \mathbf{b}^* d\mathcal{v}^* \tag{2.16}$$

where $d\mathcal{v}^*$ denotes the element of volume. In addition, let the outward unit normal vector at a material point on the boundary $\partial\mathcal{P}^*$ of the part $\mathcal{P}^*$ at time t be denoted by $\mathbf{n}^*$ and be given by

$$\mathbf{n}^* = n^*_i \mathbf{g}^{*i} = n^{*i} \mathbf{g}^*_i \tag{2.17}$$

Let $\mathbf{t}^* = \mathbf{t}^*(\eta^i,t;\mathbf{n}^*)$ be defined for the material points on the boundary $\partial\mathcal{P}^*$ at time t. The vector $\mathbf{t}^*$ is called the *contact force* or the *stress vector* acting on the part $\mathcal{P}^*$ of $\mathcal{B}^*$. The *resultant contact force* exerted on the part $\mathcal{P}^*$ at time t is then defined by

$$\mathbf{F}_c^* = \int_{\partial \mathcal{P}^*} \mathbf{t}^*(\eta^i,t;\mathbf{n}^*)da^* \tag{2.18}$$

where da^* is the element of area whose outward unit normal is $\mathbf{n}^*$. Moreover, we assume the existence of a specific internal energy density $\varepsilon^* = \varepsilon^*(\eta^i,t)$ per unit mass ρ^*.

In terms of the above definitions of the various field quantities, with reference to the present configuration and within the context of the classical (nonpolar) continuum mechanics, the conservation laws in the context of purely mechanical theory are given by

$$\begin{aligned}
&a : \frac{d}{dt}\int_{\mathcal{P}^*} \rho^* dv^* = 0 \\
&b : \frac{d}{dt}\int_{\mathcal{P}^*} \rho^* \mathbf{v}^* dv^* = \int_{\mathcal{P}^*} \rho^* \mathbf{b}^* dv^* + \int_{\partial\mathcal{P}^*} \mathbf{t}^* da^* \\
&c : \frac{d}{dt}\int_{\mathcal{P}^*} \rho^* \mathbf{p}^* \times \mathbf{v}^*\, dv^* = \int_{\mathcal{P}^*} \rho^* \mathbf{p}^* \times \mathbf{b}^* dv^* + \int_{\partial\mathcal{P}^*} \mathbf{p}^* \times \mathbf{t}^*\, da^* \\
&d : \frac{d}{dt}\int_{\mathcal{P}^*} \rho^* (\varepsilon^* + \kappa^*) dv = \int_{\mathcal{P}^*} \rho^* (\mathbf{b}^* \times \mathbf{v}^* dv^*) + \int_{\partial\mathcal{P}^*} \mathbf{t}^* \cdot \mathbf{v}^* da^*
\end{aligned} \tag{2.19}$$

where κ^* denotes the kinetic energy per unit mass ρ^* and has the form

$$\kappa^* = \tfrac{1}{2}\, \mathbf{v}^* \cdot \mathbf{v}^* \tag{2.20}$$

Equations $(2.19)_a$ to $(2.19)_d$ represent mathematical statements of conservation of mass, conservation of linear momentum, conservation of moment of momentum, and conservation of energy, respectively.

Under suitable continuity assumptions, the principle of linear momentum and that of moment of momentum imply the existence of a tensor field $\tau^{*ij} = \tau^{*ij}(\eta^k,t)$ such that

$$\begin{aligned}
&\mathbf{t}^* = \frac{\mathbf{T}^{*i}\, n_i^*}{g^{*1/2}} = \tau^{*ij}\, n_i^*\, \mathbf{g}_j^* \\
&\mathbf{T}^{*i} = g^{*1/2}\, \tau^{*ij}\, \mathbf{g}_j^* = g^{*1/2}\, \tau_j^{*i}\, \mathbf{g}^{*j}
\end{aligned} \tag{2.21}$$

Moreover, with the help of (2.21), the transport theorem, and the divergence theorem, the balance laws $(2.19)_a$ and $(2.19)_b$ can be reduced to the Cauchy equations of motion, i.e.,

$$\mathbf{T}^{i}{}_{,i} + \rho^* \mathbf{b}^* g^{*1/2} = \rho^* \mathbf{c}^* g^{*1/2}$$
$$\mathbf{g}_i^* \times \mathbf{T}^{*i} = 0 \tag{2.22}$$

where $\mathbf{c}^*$ is the acceleration vector, i.e.,

$$\mathbf{c}^* = \dot{\mathbf{v}}^* \tag{2.23}$$

and where a superposed dot is the material time derivative with respect to t hold η^i fixed. In (2.21) and (2.22) τ^{*ij} and $\tau^{*i}{}_j$ are the contravariant and mixed components of the stress tensor and a comma denotes partial differentiation with respect to η^i. Moreover, with the use of the divergence theorem and the equations of motion, it can be shown that $(2.19)_d$ reduces to

$$\rho^* g^{*1/2} \dot{\varepsilon}^* = \mathbf{T}^{*i} \cdot \mathbf{v}^*_{,i} \quad , \quad \rho^* \dot{\varepsilon}^* = \tau^{*ij} \dot{\gamma}^*_{ij} \tag{2.24}$$

For an elastic body we make the constitutive assumption that

$$\varepsilon^* = \varepsilon^*(\gamma^*_{ij}) \tag{2.25}$$

together with a similar assumption for the stress tensor τ^{*ij}. In (2.25), the dependence of ε^* on the reference metric tensor is understood, although this is not shown explicitly. Making use of (2.24) and (2.25), we obtain the results[3]

$$\tau^{*ij} = \rho^* \frac{\partial \varepsilon^*}{\partial \gamma^*_{ij}} \tag{2.26}$$

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[3] In the last expression, the partial derivative is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial \varepsilon^*}{\partial \gamma^*_{ij}} + \frac{\partial \varepsilon^*}{\partial \gamma^*_{ji}}\right)$$

A material surface in $\mathcal{B}^*$ can be defined by the equation $\xi = \xi(\eta^\alpha)$. The equations resulting from (2.5) and (2.6) with $\xi = \xi(\eta^\alpha)$ represent the parametric forms of this surface in the reference and present configuration, respectively. In particular, with reference to (2.6)

$$\xi = \xi(\eta^\alpha) = \text{constant} \tag{2.27}$$

defines a one parameter family of surfaces in space each of which is assumed to be smooth and non-intersecting. Let the surface $\xi = 0$ in the present (deformed) configuration at time t be denoted by s. Any point of this surface is specified by the position vector $\mathbf{r}$, relative to the same fixed origin 0 in the 3-dimensional space, and we have

$$\mathbf{r} = \mathbf{r}(\eta^\alpha,t) = \mathbf{p}^*(\eta^\alpha,0,t) \tag{2.28}$$

Let $\mathbf{a}_\alpha$ denote the base vectors along the η^α-curves on the surface s. Moreover, let $\mathbf{a}_3 = \mathbf{a}_3(\eta^\alpha,t)$ be the unit normal to s. We recall the results

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \eta^\alpha} = \mathbf{g}_\alpha^*(\eta^\gamma,0,t) \ , \tag{2.29}$$

$$\mathbf{a}^\alpha \cdot \mathbf{a}_3 = 0 \ , \ \mathbf{a}_3 \cdot \mathbf{a}_3 = 1 \ , \ \mathbf{a}_3 = \mathbf{a}^3 \ , \ [\mathbf{a}_1\, \mathbf{a}_2\, \mathbf{a}_3] \neq 0 \ . \tag{2.30}$$

We also recall the formulae

$$\begin{gathered} a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \ , \ a = \det(a_{\alpha\beta}) \ , \ \mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta \ , \\ \mathbf{a}^\alpha \cdot \mathbf{a}^\beta = a^{\alpha\beta} \ , \ a^{\alpha\gamma}a_{\gamma\beta} = \delta^\alpha{}_\beta \ , \end{gathered} \tag{2.31}$$

$$b_{\alpha\beta} = b_{\beta\alpha} = -\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta} = \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta} \tag{2.32}$$

$$\mathbf{a}_{\alpha|\beta} = b_{\alpha\beta}\mathbf{a}_3 \ , \ \mathbf{a}_{3,\alpha} = -b^\gamma{}_\alpha \mathbf{a}_\gamma \ , \ b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta} \tag{2.33}$$

where $a_{\alpha\beta}$ is the metric tensor of the surface and $b_{\alpha\beta}$ are the coefficients of the second fundamental form of the surface. We recall that the three equations given by (2.33) are the formulae

of Gauss, Weingarten and the Mainardi-Codazzi, respectively.

Considering expression (2.28), we recall that **r** is the position vector of a typical point of the surface s, i.e., the material surface $\xi = 0$ in the present configuration of the body $\mathcal{B}^*$ at time t. Let the corresponding surface (i.e., $\xi = 0$) in the reference configuration be denoted by $\mathcal{S}$. Any point of this surface in the reference configuration, is specified by:

$$\mathbf{R} = \mathbf{R}(\eta^\alpha) = \mathbf{P}(\eta^\alpha, 0) \tag{2.34}$$

It should be clear that if the reference configuration of $\mathcal{B}^*$ is chosen to be the initial configuration at time t = 0, then we will have

$$\mathbf{R} = \mathbf{R}(\eta^\alpha) = \mathbf{r}(\eta^\alpha, 0) \tag{2.35}$$

Let A_α be the base vectors along the coordinate curves on the surface $\mathcal{S}$. Then we have

$$A_\alpha = \frac{\partial \mathbf{R}}{\partial \eta^\alpha} = G_\alpha(\eta^\gamma, 0) \tag{2.36}$$

and

$$A_\alpha \cdot A_3 = 0 \ , \ A_3 \cdot A_3 = 1 \ , \ A_3 = A^3 \ , \ [A_1\, A_2\, A_3] \neq 0 \tag{2.37}$$

where A_3 is the unit normal to $\mathcal{S}$. The duals of the relations (2.31) to (2.33) are given by

$$\begin{gathered} A_{\alpha\beta} = A_\alpha \cdot A_\beta \ , \ A = \det(A_{\alpha\beta}) \ , \ A^\alpha = A^{\alpha\beta} A_\beta \ , \\ A^\alpha \cdot A^\beta = A^{\alpha\beta} \ , \ A^{\alpha\gamma} A_{\gamma\beta} = \delta^\alpha{}_\beta \end{gathered} \tag{2.38}$$

$$B_{\alpha\beta} = B_{\alpha\beta} = -A_\alpha \cdot A_{3,\beta} = A_3 \cdot A_{\alpha,\beta} \tag{2.39}$$

$$A_{\alpha|\beta} = B_{\alpha\beta} A_3 \ , \ A_{3,\alpha} = -B^\gamma{}_\alpha A_\gamma \ , \ B_{\alpha\beta|\gamma} = B_{\alpha\gamma|\beta} \tag{2.40}$$

Before closing this section we recall basic jump conditions in the context of three-dimensional classical continuum mechanics. Suppose that at time t an arbitrary material volume

of the body occupies a part $\mathcal{P}^*$ bounded by a closed surface $\partial\mathcal{P}^*$. Let $\mathcal{P}^*$ be divided into two regions $\mathcal{P}_1^*$, $\mathcal{P}_2^*$ separated by a moving surface $\sigma(t)$, and let $\partial\mathcal{P}^{*\prime}, \partial\mathcal{P}^{*\prime\prime}$ denote the portions of the surface $\partial\mathcal{P}^*$ which form parts of the boundaries $\partial\mathcal{P}_1^*$ and $\partial\mathcal{P}_2^*$ such that

$$\begin{gathered}
\partial\mathcal{P}^{*\prime} = \partial\mathcal{P}_1^* \cap \partial\mathcal{P}^* \quad , \quad \partial\mathcal{P}^{*\prime\prime}\, \partial\mathcal{P}_2^* \cap \partial\mathcal{P}^* \\
\mathcal{P}^* = \mathcal{P}_1^* \cup \mathcal{P}_2^* \quad , \quad \partial\mathcal{P}^* = \partial\mathcal{P}^{*\prime} \cup \partial\mathcal{P}^{*\prime\prime} \\
\partial\mathcal{P}_1^* = \partial\mathcal{P}^{*\prime} \cup \sigma(t) \quad , \quad \partial\mathcal{P}_2^* = \partial\mathcal{P}^{*\prime\prime} \cup \sigma(t)
\end{gathered} \tag{2.41}$$

Let the velocity of the surface $\sigma(t)$ along its outward normal, when σ is regarded as part of the boundary $\partial\mathcal{P}_1^*$, be denoted by u_n. Then, $-u_n$ is the normal velocity of σ when this surface is regarded as part of the boundary of $\mathcal{P}_2^*$. Let ψ be any function which takes different values ψ_1 and ψ_2 on either side of σ in the regions $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$, respectively. We adapt the notation $[\![\ ]\!]$ to indicate the difference of ψ_2 and ψ_1 and write

$$[\![\psi]\!] = \psi_2 - \psi_1 \tag{2.42}$$

We also adopt the notations

$$w_{1n} = v_{1n} - u_n \ , \ w_{2n} = v_{2n} - u_n \tag{2.43}$$

where v_{1n} and v_{2n} are the velocities of the material points in the regions $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ along the normal to σ, respectively. Then within the context of classical three-dimensional continuum mechanics, we recall the following jump conditions

$$\begin{aligned}
&a : [\![\rho^* w_n^*]\!] = 0 \\
&b : [\![\rho^* \mathbf{v}^* w_n^* - \mathbf{t}^*]\!] = 0 \\
&c : \text{no new equation} \\
&d : [\![\rho^*(\varepsilon^* + k^*) w_n^* - \mathbf{t}^* \cdot \mathbf{v}^*]\!] = 0
\end{aligned} \tag{2.44}$$

3. Modeling of a composite laminate

To begin with, the continuum itself is a model representing an idealized body in some sense. We may recall that the continuum model (in classical mechanics) is intended to represent phenomena in nature which appear at a scale larger than the interatomic distances. From such intuitive notions the well defined classical field theories of mechanics have been constructed and the "macroscopic" behavior of the general medium in question has been successfully studied. In the context of classical continuum mechanics a body is thought of as a set of particles (material points), say x. Each material point has a distinct identity and occupies at each instant of time t an exclusive place in a Euclidean three-dimensional space, so that one can identfy each material point x with its place (i.e., the position vector from a fixed reference point) in the space. It is implied that no more interesting information would be perceived by a finer observation of material points. Hence, "microscopic" details, if any, are discarded.

For a large class of bodies, these preconceptions are justified, but there are also cases when a closer look at a material point reveals some microscopic order and that at least partial information of interest could be extracted by considering the effect of the microscopic order. It is therefore desirable to construct continuum theories that in some fashion incorporate the effect of the microstructure while enjoying, if possible, to some extent the level of generality available in the classical continuum mechanics. There are different types of materials that exhibit microstructural behavior. One class of such materials is composites, i.e., bodies in which two or more substances are combined in a specific geometrical fashion to produce a new material with mechanical properties different from those of the individual constituents. Roughly speaking, a continuum with microstructure is a continuum whose properties and behavior are affected by the local deformations of the material points in any of its volume elements.

It is conceivable that equations which represent the macroscopic motion in one scale may describe motion on a scale which in some sense is microscopic. Indeed, the practical analysis of the mechanical response of composite bodies involves analytical studies on two levels of

abstraction. These areas of investigation are known as *micromechanics* and *macromechanics.* In micromechanics, one attempts to recognize the fine details of the material structure, i.e., a heterogeneous body, consisting of reinforcing elements, such as fibers, plies, particles, etc., embedded in a matrix material. In other words, micromechanics establishes the relation between the properties of the constituents and those of a unit composite cell. In macromechanics, on the other hand, one attempts to consider the composite body as an assembly of interacting cells, and study the overall behavior of the composite. For clarity, we emphasize that the term micromechanics does not imply studies on the atomic scale. At the same time we note that within the context of the present discussion, the physical dimension(s) involved at the microstructural level are much smaller than the physical dimension(s) involved at the macrostructural level. In what follows we confine our attention to laminated composite bodies.

We define a composite laminate as a three-dimensional continuum consisting of multiple (two or more) layers of materials which act together as a single (integral) physical entity. Here we confine our attention to laminated composites composed of alternating layers of only two materials, each of which are considered to be homogeneous. The layers are not considered to be necessarily flat and could have any type of curvature (see figure 1). We assume the thickness of each layer (ply) is much smaller than its other two dimensions and also smaller than the dimension of the composite laminate in the same direction (i.e., stack up direction). For example if θ^α are curvilinear surface coordinates of a layer (ply) and θ^3 is the third, out of surface, coordinate of the layer and the layers alternate in the direction of θ^3, the dimension of one set of alternating layers (one of each material) is much smaller in comparison to the dimension of the composite in the direction of θ^3.

In order to construct a continuum theory, we look for a (some) representative (repetitive) feature(s) within the body. For the laminated medium under consideration the most distinct representative character is the alternating feature of the layers. Hence, we choose the combination of one layer of reinforcement and one layer of matrix as a representative element (micro-

structures) for the laminated composite. We then model this representative shell-like element as a two-dimensional continuum, i.e., Cosserat (directed) surface which is a material surface together with a deformable vector field called director. Next we assume the composite laminate is composed of infinitely many of such representative elements (Cosserat surfaces) adjacent to each other.

Consider a finite three-dimensional body $\mathcal{B}$ in a Euclidean 3-space and let a set of convected coordinates θ^i $(i = 1,2,3)$ be assigned to each particle (material point) P of $\mathcal{B}$. Assume at each particle P there exists a Cosserat surface, s (i.e., a material surface together with a deformable vector field called the director) such that θ^α are the coordinates of the surface. If at each point P the Cosserat surface is now identified by a representative element (i.e., one layer of matrix together with one layer of reinforcement) of the laminated composite and if the body $\mathcal{B}$ is identified with the composite laminate itself, the model of a composite laminate with micro-structure is at hand. It is to be emphasized that in the present discussion each representative element (Cosserat surface) is itself a three dimensional shell-like body $\mathcal{B}^*$ consisting of two layers of different homogeneous materials. We also notice that the material points within each representative element $\mathcal{B}^*$ are regular particles in the sense of classical continuum mechanics while the material points of $\mathcal{B}$ are endowed not only with an "assigned mass" density but also with a "director." For clarity, we will refer to the body $\mathcal{B}$ as *composite laminate, macro-continuum* or *macro-structure* and to the body $\mathcal{B}^*$ as *representative element, micro-continuum* or *micro-structure*. Also, we will refer to particles of $\mathcal{B}$ as *macro-particles* or *composite particles* while the particles of the micro-structures will be referred to as *micro-particles* or simply particles (material points). We reiterate that parameters or variables that represent similar physical quantities in micro-body, Cosserat surface and macro-body will be designated with the same symbol but with an additional asterisk (*) and an over hat (ˆ) for the micro-body and Cosserat surface, respectively. For example, the mass density of the composite laminate will be called composite mass (or macro-mass) density and will be denoted by ρ while the mass densities of the Cosserat surface and that of the micro-structure will be designated by $\hat{\rho}$ and ρ^*, respectively. A precise definition of a shell-like

micro-structure is given in section 3.2. For now, we recall that each representative element (Cosserat surface) represents a three-dimensional body in the sense of classical continuum mechanics and its boundary consists of a lateral (normal) surface and two major (upper and lower) surfaces. We assume that at each composite particle the reference surface coincides with the lower surface of the shell-like micro-structure. Hence, each geometric point P of the body $\mathcal{B}$ is considered to coincide with a point P^* on the lower surface of the shell-like micro-structure. As it will become clear shortly, in order to adequately describe the motion of the micro-structure, we need to introduce an additional independent variable along the thickness of layering. This, roughly speaking, implies that a fourth special dimension is required for adequate description of composite laminates.

3.1. Coordinate systems for a composite laminate

At each point P of the macro-body $\mathcal{B}$ we introduce a set of convected coordinates θ^i (i = 1,2,3). Also, at each point P^* on the lower surface of the shell-like micro-structure which coincides with P we introduce another set of convected coordinates η^i (i = 1,2,3). We assume the transformation from θ^α to η^α exists, i.e.,

$$\theta^\alpha = \theta^\alpha(\eta^\beta) = \theta^\alpha(\eta^1,\eta^2) \tag{3.1}$$

and

$$\det\left(\frac{\partial\theta^\alpha}{\partial\eta^\beta}\right) \neq 0 \tag{3.2}$$

This implies the existence of a unique inverse for the above transformation. However, for η^3 we assume that it can vary independent of θ^3 and that the range of variation of η^3 is $0 \leq \eta^3 \leq \xi_2(\theta^3)$ at each point of the composite. This assumption adds an extra dimension to our classical three dimensional space. At this point we make the additional assumption that

$$\eta^\alpha = \theta^\alpha \quad (\alpha = 1,2) \tag{3.3}$$

$$\xi_2 = \varepsilon\theta^3 \quad , \quad \varepsilon \ll 1$$

The first of the above assumptions is for convenience (not necessary) while the second one is needed since the thickness of a representative element (micro-structure) is considered to be much smaller than the dimension(s) of the composite laminate (macro-structure). As before, for convenience we set $\eta^3 = \xi$ and adopt the notation $\eta^i = \{\eta^\alpha, \xi\}$. For the sake of clarity we note that the covariant and the contravariant base vectors in the coordinate system η^i ($i = 1,2,3$) are denoted by $\mathbf{g}_i^*$ and $\mathbf{g}^{*i}$, respectively. In this connection we recall the formulae

$$\begin{gathered}
\mathbf{g}_i^* = \frac{\partial \mathbf{p}^*}{\partial \eta^i} \quad , \quad g_{ij}^* = \mathbf{g}_i^* \cdot \mathbf{g}_j^* \quad , \quad g^* = \det(g_{ij}^*) \quad , \quad g^{*1/2} = [\mathbf{g}_1^* \, \mathbf{g}_2^* \, \mathbf{g}_3^*] > 0 \ , \\
\mathbf{g}^{*i} = g^{*ij}\mathbf{g}_j^* \quad , \quad \mathbf{g}^{*i} \cdot \mathbf{g}^{*j} = g^{*ij} \quad , \quad \mathbf{g}^{*i} \cdot \mathbf{g}_j^* = \delta^i{}_j
\end{gathered} \tag{3.4}$$

where $\mathbf{p}^* = \mathbf{p}^*(\eta^i)$ is a position vector in η^i coordinate system. The covariant and the contravariant base vectors in the coordinate system θ^i ($i = 1,2,3$) are denoted by $\mathbf{g}_i$ and $\mathbf{g}^i$, respectively. The duals of (3.4) in θ^i ($i = 1,2,3$) coordinate system are given by

$$\begin{gathered}
\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \eta^i} \quad , \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \quad , \quad g = \det(g_{ij}) \quad , \quad g^{1/2} = [\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3] \neq 0 \\
\mathbf{g}^i = g^{ij}\mathbf{g}_j \quad , \quad \mathbf{g}^i \cdot \mathbf{g}^j = g^{ij} \quad , \quad \mathbf{g}^i \cdot \mathbf{g}_j = \delta^i{}_j
\end{gathered} \tag{3.5}$$

where $\mathbf{r} = \mathbf{r}(\theta^i)$ is a position vector in θ^i coordinate system.

3.2. Definition of a shell-like representative element (micro-structure)

Within the context of three-dimensional classical continuum mechanics, consider a body $\mathcal{B}^*$ in the present configuration and let its boundary be a closed surface, denoted by $\partial\mathcal{B}^*$, and be composed of the following material surfaces[4]:

[4] Most of the literature dealing with shell-like bodies choose a reference surface between the top and the bottom faces of the shell-like body. Here it seems more convenient to choose the bottom surface as our reference surface. The importance of choosing the top (or bottom) surface as a reference surface was brought up in a different context by Naghdi [1975] in relation to contact problems of shells.

a) The material surfaces

$$\begin{aligned} s_0 &: \ \xi = 0 \\ & \qquad\qquad 0 < \xi_2 \\ s_2 &: \ \xi = \xi_2(\eta^\alpha) \end{aligned} \tag{3.6}$$

b) The material surface

$$s_l: \ f(\eta^\alpha) = 0 \tag{3.7}$$

such that ξ = const. are closed smooth curves on the surface (3.7). We also consider a material surface of the form

$$s_1: \ \xi = \xi_1(\eta^\alpha) \qquad 0 < \xi_1 < \xi_2 \tag{3.8}$$

lying entirely between s_0 and s_2. From now on we will refer to surfaces defined above as follows.

a) s_0 : bottom face (lower major surface) of the micro-structure.

b) s_1 : interface (middle major surface) of the micro-structure.

c) s_2 : top face (upper major surface) of the micro-structure.

d) s_l : lateral (major) surface or normal surface of the micro-structure.

We recall that since $\eta^i = \{\eta^\alpha, \xi\}$ are defined as convected coordinates, the material surfaces (3.6) and (3.7) will have the same parametric representation in all configurations. In general ξ_1 and ξ_2 are functions of the surface coordinate η^α but in special cases they may be constants. We assume the surfaces s_0, s_1 and s_2 do not intersect themselves, or each other. This implies the condition $(3.8)_2$ and $g^* \neq 0$. The surface s_1 is not necessarily midway between the bounding surfaces s_0 and s_2. Such a three dimensional body $\mathcal{B}^*$ as characterized above and depicted in figure

2, is called a shell-like representative element or a shell-like micro-structure if the dimension of the body along the normals to the surface s_o, called the *height* of the micro-structure, is much smaller in comparison to its other two dimensions or a characteristic length of the surface s_o.

Considering our description of the body $\mathcal{B}^*$, we may note that $\mathcal{B}^*$ consists of two distinct parts $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ as defined below.

a) Part $\mathcal{B}_1^*$, a shell-like body bounded by the major surfaces s_0 and s_1 and by a lateral surface s_{l_1} which is the portion of the surface s_l bounded by its intersections with s_o and s_1.

b) Part $\mathcal{B}_2^*$, a shell-like body bounded by the major surfaces s_1 and s_2 and by a lateral surface s_{l_2} which is the portion of the surface s_l bounded by its intersections with s_1 and s_2.

In view of (a) and (b) above, we have

$$\mathcal{B}^* = \mathcal{B}_1^* \cup B_2^*$$

$$s_l = s_{l_1} \cup s_{l_2} \tag{3.9}$$

We assume that $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ consist of two different materials which are perfectly bonded at their interface surface, namely the surface $s_1 : \xi = \xi_1$. We will designate the physical quantities associated with $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ with subscripts 1 and 2, respectively. For example, the mass densities of $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ will be designated by ρ_1^* and ρ_2^*, respectively. It is clear that the physical quantities associated with the body $\mathcal{B}^*$ may have a jump across the surface $s_1 :\ \xi = \xi_1$.

Let $\rho^*(\eta^\alpha,\xi,t)$ and $\rho_o^*(\eta^\alpha,\xi)$ be the mass densities of $\mathcal{B}^*$ in the deformed and reference configurations, respectively. Then the conservation of mass (in three dimensions) implies

$$\rho_\alpha^* g^{*1/2} = \rho_{o\alpha}^* G^{*1/2} \quad (\alpha = 1,2) \tag{3.10}$$

We define the micro-structure mass density, per unit area of s_o, at time t in the present configuration by the expression

$$\hat{\rho}\, a^{1/2} = \int_o^{\xi_2} \rho^* g^{*1/2} d\xi$$
$$\hat{\rho} = \hat{\rho}(\eta^\alpha, t) \tag{3.11}$$

where $\hat{\rho}$ denotes the mass density and a is $\det(a_{\alpha\beta})$ of the surface s_o. In view of our description of the body $\mathcal{B}^*$, we have

$$\hat{\rho}\, a^{1/2} = \int_o^{\xi_2} \rho^*\, g^{*1/2}\, d\xi = \int_o^{\xi_1} \rho_1\, g^{*1/2}\, d\xi + \int_{\xi_1}^{\xi_2} \rho_2^*\, g^{*1/2}\, d\xi \tag{3.12}$$

Since the quantities $\rho_1^*\, g^{*1/2}$ and $\rho_2^*\, g^{*1/2}$ are independent of time, it follows that $\hat{\rho}\, a^{1/2}$ is also independent of time, although both $\hat{\rho}$ and a may depend on t. The total mass of an arbitrary part $\mathcal{P}^*$ of the body $\mathcal{B}^*$ (composed of parts $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ of $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$, respectively) bounded by the surface $(3.6)_{1,2}$ and a surface of the form (3.7) may be expressed by

$$\mathcal{M}^* = \int_{\mathcal{P}} \rho^* dv^* = \int_{\hat{\eta}_1} \int_{\hat{\eta}_2} \int_o^{\xi_2} \rho^*\, g^{*1/2}\, d\eta^1\, d\eta^2\, d\xi \tag{3.13a}$$

or

$$\mathcal{M}^* = \mathcal{M}_1^* + \mathcal{M}_2^* = \int_{\hat{\eta}_1} \int_{\hat{\eta}_2} \hat{\rho}\, a^{1/2}\, d\eta^1 d\eta^2 = \int_{\hat{\mathcal{P}}} \hat{\rho}\, d\hat{a} \tag{3.13b}$$

where

$$\mathcal{M}_1^* = \int_{\hat{\eta}_1} \int_{\hat{\eta}_2} \int_o^{\xi_1} \rho_1^*\, g^{*1/2}\, d\eta^1 d\eta^2 d\xi$$
$$\mathcal{M}_2^* = \int_{\hat{\eta}_1} \int_{\hat{\eta}_2} \int_o^{\xi_2} \rho_2^*\, g^{*1/2}\, d\eta^1 d\eta^2 d\xi \tag{3.14}$$

where $\hat{\mathcal{P}}$ denotes an arbitrary part of the surface $s_o : \xi = 0$ which corresponds to $\mathcal{P}^*$ and $\hat{\eta}_1, \hat{\eta}_2$

denote the applicable ranges of integration for the coordinates η^1 and η^2, respectively. Also, in obtaining (3.13b) we have made use of (3.12) and the following formula:

$$d\mathcal{v}^* = (\mathbf{g}_1^* \times \mathbf{g}_2^*) \cdot \mathbf{g}_3^* \, d\eta^1 d\eta^2 d\xi = g^{*1/2} d\eta^1 d\eta^2 d\xi \tag{3.15}$$

$$d\mathcal{a} = (\mathbf{a}_1 \times \mathbf{a}_2) \cdot \mathbf{a}_3 \, d\eta^1 d\eta^2 = a^{1/2} \, d\eta^1 d\eta^2 \tag{3.16}$$

For later use we define the following quantities

$$\hat{\rho} a^{1/2} = \hat{\lambda} = \int_o^{\xi_2} \lambda^* \, d\xi \ , \ \lambda^* = \rho^* g^{*1/2} \tag{3.17}$$

and

$$\hat{\rho} a^{1/2} y^\alpha = \hat{\lambda} y^\alpha = \int_o^{\xi_2} \lambda^* \xi^\alpha d\xi \ , \qquad (\alpha = 1,2) \tag{3.18}$$

In view of (3.12), we may rewrite (3.17) as

$$\hat{\rho} a^{1/2} = \hat{\lambda} = \hat{\lambda}_1 + \hat{\lambda}_2 \tag{3.19}$$

where

$$\begin{aligned} \hat{\lambda}_1 &= \int_o^{\xi_1} \lambda^* d\xi = \int_o^{\xi_1} \rho_1^* \, g^{*1/2} d\xi \\ \hat{\lambda}_2 &= \int_{\xi_1}^{\xi_2} \lambda^* d\xi = \int_{\xi_1}^{\xi_2} \rho_2^* \, g^{*1/2} d\xi \end{aligned} \tag{3.20}$$

Also, expression (3.18) may be rewritten

$$\hat{\rho} a^{1/2} y^\alpha = \hat{\lambda} \, y^\alpha = \hat{\lambda}_1 y^\alpha + \hat{\lambda}_2 y^\alpha \tag{3.21}$$

where

$$\hat{\lambda}_1 y^\alpha = \int_o^{\xi_1} \lambda^* \xi^\alpha d\xi = \int_o^{\xi_1} \rho_1^* g^{*1/2} \xi^\alpha d\xi$$

$$\hat{\lambda}_2 y^\alpha = \int_{\xi_1}^{\xi_2} \lambda^* \xi^\alpha d\xi = \int_{\xi_1}^{\xi_2} \rho_2^* g^{*1/2} \xi^\alpha d\xi \tag{3.22}$$

This completes our description of a shell-like micro-structure (representative element), namely a three dimensional body $\mathcal{B}^*$ composed of two shell-like bodies $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ such that

$$\mathcal{B}^* = \mathcal{B}_1^* \cup \mathcal{B}_2^*$$

where $\mathcal{B}^*$ is bounded by the surfaces $(3.6)_{1,2}$ and (3.7), $\mathcal{B}_1^*$ is bounded by the surfaces $(3.6)_1$, (3.8) and (3.7), $\mathcal{B}_2^*$ is bounded by the surfaces (3.8), $(3.6)_2$ and (3.7). We assume $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ are perfectly bonded together at the surface (3.8).

We will refer to the duals of the surface s_0, s_1, s_2 in the reference configuration by $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$, respectively. We also note that the duals of (3.11) and (3.12) in the reference configuration are as follows:

$$\hat{\rho}_o A^{1/2} = \int_o^{\xi_2} \rho_o^* \, G^{*1/2} \, d\xi \tag{3.23}$$

and we have

$$\hat{\rho}_o \, A^{1/2} = \hat{\rho} \, a^{1/2} \tag{3.24}$$

Also,

$$\hat{\rho}_o \, A^{1/2} = \int_o^{\xi_2} \rho_o^* \, G^{*1/2} \, d\xi = \int_o^{\xi_1} \rho_{o1}^* \, G^{*1/2} \, d\xi + \int_{\xi_1}^{\xi_2} \rho_{o2}^* \, G^{*1/2} \, d\xi \tag{3.25}$$

and we have

$$\hat{\rho}_{o1} \, A^{1/2} = \hat{\rho}_1 \, a^{1/2} \ , \ \hat{\rho}_{o2} \, A^{1/2} = \hat{\rho}_2 \, a^{1/2} \tag{3.26}$$

in view of (3.10).

4. Kinematics of micro- and macro-structures

We begin our development of the kinematical results by assuming that the position vector of a particle P^* of a thin representative element (micro-structure), i.e., $\mathbf{p}^*(\eta^\alpha,\xi,\theta^3,t)$ in the present configuration has the form

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,\theta^3,t) + \xi\mathbf{d}(\eta^\alpha,\theta^3,t) \tag{4.1}$$

The above is a special assumption which is regarded to be valid for thin shells[5]. The dual of (4.1) in a reference configuration is given by

$$\mathbf{P}^* = \mathbf{R}^*(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3) \tag{4.2}$$

If the reference configuration is taken to be the initial configuration at time $t = 0$,

$$\mathbf{p}^*(\eta^\alpha,\xi,\theta^3,0) = \mathbf{r}(\eta^\alpha,\theta^3,0) + \xi\mathbf{d}(\eta^\alpha,\theta^3,0) = \mathbf{R}(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3) = \mathbf{P}(\eta^\alpha,\xi,\theta^3) \tag{4.3}$$

The velocity vector $\mathbf{v}^*$ of the three-dimensional shell-like micro-structure at time t is given by

$$\mathbf{v}^* = \dot{\mathbf{p}}^*(\eta^\alpha,\xi,\theta^3,t) \tag{4.4}$$

where a superposed dot denotes the material time derivative, holding η^i and θ^i fixed. From (4.1) and (4.4) we obtain

$$\mathbf{v}^* = \mathbf{v} + \xi\mathbf{w} \tag{4.5}$$

where

$$\mathbf{v} = \dot{\mathbf{r}} \ , \ \mathbf{w} = \dot{\mathbf{d}} \tag{4.6}$$

[5] In a more general approach, we may begin the kinematical development by assuming that $\mathbf{p}(\eta^\alpha,\xi,t)$ is an analytical function of ξ in the region $0 < \xi < \xi_2$ and can be represented as [Naghdi, 1972, section 7]

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,\theta^3,t) + \sum_{N=1}^{\infty} \xi^n \, \mathbf{d}_N(\theta^\alpha,\theta^3,t)$$

This generality is not needed for our present purposes and we therefore adhere to the assumption (6.1).

From (4.1) and (2.7) we have

$$\mathbf{g}_\alpha^* = \mathbf{a}_\alpha + \xi \frac{\partial \mathbf{d}_{,\alpha}}{\partial \eta^\alpha} \ , \ \mathbf{g}_3^* = \mathbf{d} \tag{4.7}$$

where $\mathbf{a}_\alpha$ are the surface base vector of the surface s_o. The base vectors $\mathbf{g}_i^*(\eta^\alpha,\xi,\theta^3,t)$ in (4.7) when evaluated on the surface $s_o : \xi = 0$ reduce to

$$\begin{aligned} \mathbf{g}_\alpha^*(\eta^\gamma,0,\theta^3,t) &= \mathbf{a}_\alpha(\eta^\gamma,\theta^3,t) \\ \mathbf{g}_3^*(\eta^\gamma,0,\theta^3,t) &= \mathbf{d}(\eta^\gamma,\theta^3,t) \end{aligned} \tag{4.8}$$

where $\mathbf{g}_i^*$ satisfy the condition

$$[\mathbf{g}_1^* \ \mathbf{g}_2^* \ \mathbf{g}_3^*] > 0 \tag{4.9}$$

This restriction holds for all time and values of $\eta^i = \{\eta^\alpha,\xi\}$ and θ^3. In particular, it is valid for $\xi = 0$ so that by (4.9) we also have

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{d}] \neq 0 \tag{4.10}$$

this condition implies that the director $\mathbf{d}$ cannot be tangent to the surface s_o. In addition, we have

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \theta^i} \tag{4.11}$$

and

$$[\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3] > 0 \tag{4.12}$$

From (4.1), (4.3) and (4.11), it is clear that

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha \tag{4.13}$$

We recall that the director **d** is a three-dimensional vector and it can be written as

$$\mathbf{d} = d_i \mathbf{g}^i = d^i \mathbf{g}_i \quad , \quad d_i = \mathbf{g}_i \cdot \mathbf{d} \quad , \quad d^i = g^{ij} d_j \tag{4.14}$$

where d_i and d^i denote the covariant and contravariant components of **d** referred to $\mathbf{g}^i$ and $\mathbf{g}_i$, respectively. The gradient of the director **d** with respect to θ^i may be obtained as follows:

$$\mathbf{d}_{,i} = (d^j \mathbf{g}_j)_{,i} = d^j{}_{,i} \mathbf{g}_j + d^j \mathbf{g}_{j,i} = d^j{}_{,i} \mathbf{g}_j + d^j \{{}_i{}^k{}_j\} \mathbf{g}_k = d^j \mathbf{g}_j d^j{}_{|i} \mathbf{g}_j \tag{4.15}$$

where { } stands for the Christoffel symbol of the second kind and a vertical bar (|) denotes covariant differentiation with respect to g_{ij}. For convenience we introduce the notations

$$\lambda_{ij} = \mathbf{g}_i \cdot \mathbf{d}_{,j} = d_i|_j$$
$$\lambda^i{}_j = \mathbf{g}^i \cdot \mathbf{d}_{,j} = d^i|_j \tag{4.16}$$

From (4.16) it is clear that

$$\mathbf{d}_{,i} = \lambda_{ji} \mathbf{g}^j = \lambda^j{}_i \mathbf{g}_j \quad , \quad \lambda^i{}_j = g^{ik} \lambda_{kj} \tag{4.17}$$

Consider now the velocity vector **v** which can be written in the form

$$\mathbf{v} = v^i \mathbf{g}_i = v_i \mathbf{g}^i \quad , \quad v_i = \mathbf{g}_i \cdot \mathbf{v} \quad , \quad v^i = g^{ij} v_j \tag{4.18}$$

Since the coordinates θ^i are convected, it follows that

$$\mathbf{v}_{,i} = \mathbf{v}_{|i} = \dot{\mathbf{g}}_i \tag{4.19}$$

Following the same procedure used in (4.15), we can reduce (4.10) to

$$\mathbf{v}_{,i} = (v^j \mathbf{g}_j)_{,i} = v^j{}_{,i} \mathbf{g}_j + v^j \mathbf{g}_{j,i} = v^j{}_{,i} \mathbf{g}_j + \{{}_i{}^k{}_j\} v^j \mathbf{g}_k = v^j{}_{|i} \mathbf{g}_j \tag{4.20}$$

We now introduce the notations

$$v_{ij} = \mathbf{g}_i \cdot \mathbf{v}_{,j} = v_{i|j}$$
$$v^i{}_j = \mathbf{g}^i \cdot \mathbf{v}_{,j} = v^i{}_{|j} \tag{4.21}$$

From (4.21) it is clear that

$$\mathbf{v}_{,i} = v_{ji}\mathbf{g}^j = v^j{}_i\mathbf{g}_j \quad , \quad v^i{}_j = g^{ik}v_{k|j} \tag{4.22}$$

We observe that both λ_{ij} and v_{ij} represent the covariant derivative of vector components and hence transform as components of second order covariant tensors.

Since v_{ij} is a second order covariant tensor, we may decompose it into its symmetric and its skew-symmetric parts, i.e.,

$$v_{ij} = v_{(ij)} + v_{[ij]} = \eta_{ij} + \omega_{ij} \tag{4.23}$$

where

$$\eta_{ij} = v_{(ij)} = \frac{1}{2}(v_{ij} + v_{ji}) \quad , \quad \omega_{ij} = v_{[ij]} = \frac{1}{2}(v_{ij} - v_{ji}) \tag{4.24}$$

represent the symmetric and the skew-symmetric parts of v_{ij}, respectively. From (4.24), after making use of (4.21) and (4.23), we have

$$\eta_{ij} = \frac{1}{2}(v_{ij} + v_{ji}) = \frac{1}{2}(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j + \mathbf{g}_j \cdot \dot{\mathbf{g}}_i) = \frac{1}{2}\,\dot{\overline{(\mathbf{g}_i \cdot \mathbf{g}_j)}} = \frac{1}{2}\,\dot{g}_{ij} = \eta_{ji} \tag{4.25}$$

and

$$\omega_{ij} = \frac{1}{2}(v_{ij} - v_{ji}) = \frac{1}{2}(\mathbf{g}_i \cdot \dot{\mathbf{g}}_j - \mathbf{g}_j \cdot \dot{\mathbf{g}}_i) = -\omega_{ji} \tag{4.26}$$

Also, in view of (4.20) and (4.24), we may express $\dot{\mathbf{g}}_i$ in the form

$$\dot{\mathbf{g}}_i = \mathbf{v}_{,i} = (\eta_{ki} + \omega_{ki})\mathbf{g}^k \tag{4.27}$$

Moreover, the time rate of change of the determinant of g_{ij}, i.e., g is obtained as follows

$$\dot{g} = \overline{\det(g_{ij})}^{\cdot} = \frac{\partial}{\partial g_{kl}} (\det(g_{kj}))\dot{g}_{kl} = g g^{ij} \dot{g}_{ij} \tag{4.28}$$

Also, by making use of the relation $g^{ij} g_{kj} = \delta^i{}_j$, we obtain an expression for $\dot{g}^{ij}$

$$\dot{g}^{ij} = -\, g^{ik} g^{jl} \dot{g}_{kl} \tag{4.29}$$

Next, we obtain an expression for the director velocity **w**. Thus, we write

$$\mathbf{w} = \dot{\mathbf{d}} = w_k \mathbf{g}^k = w^k \mathbf{g}_k = \overline{(d_i \mathbf{g}^i)}^{\cdot} = \dot{d}_k \mathbf{g}^k + d^i(\omega_{ki} - \eta_{ki})\mathbf{g}^k \tag{4.30}$$

where in obtaining (4.30) we have made use of (4.27) and (4.29). The gradient of the director velocity is obtained in a similar manner:

$$\mathbf{w}_{,i} = \dot{\mathbf{d}}_{,i} = \overline{(\mathbf{d}_k \mathbf{g}^k)_{,i}}^{\cdot} = \overline{(\lambda_{ki} \mathbf{g}^k)}^{\cdot} = \dot{\lambda}_{ki} \mathbf{g}^k + \lambda^j{}_i(\omega_{kj} - \eta_{kj})\mathbf{g}^k \tag{4.31}$$

The dual of expressions (4.7) to (4.17) in the reference configuration follows from (4.2) in a similar manner and is given by:

$$\mathbf{G}^*_\alpha = \mathbf{A}_\alpha + \xi \mathbf{D}_{,\alpha} \quad , \quad \mathbf{G}^* = \mathbf{D} \tag{4.32}$$

$$\begin{aligned} \mathbf{G}^*_\alpha(\eta^\gamma,0,\theta^3) &= \mathbf{A}_\alpha(\eta^\gamma,\theta^3,t) \\ \mathbf{G}^*_3(\eta^\gamma,0,\theta^3) &= \mathbf{D}(\eta^\gamma,\theta^3,t) \end{aligned} \tag{4.33}$$

where $\mathbf{G}^*_i$, $\mathbf{D}$ satisfy the conditions

$$[\mathbf{G}^*_1 \; \mathbf{G}^*_2 \; \mathbf{G}^*_3] \neq 0 \tag{4.34}$$

and

$$[\mathbf{G}^*_1 \; \mathbf{G}^*_2 \; \mathbf{D}] \neq 0 \tag{4.35}$$

In addition, we have

$$\mathbf{G}_i = \frac{\partial \mathbf{R}}{\partial \theta^i} \tag{4.36}$$

$$[\mathbf{G}_1\mathbf{G}_2\mathbf{G}_3] > 0 \tag{4.37}$$

and

$$\mathbf{G}^\alpha = \mathbf{A}_\alpha \tag{4.38}$$

Moreover,

$$\begin{gathered} \mathbf{D} = D_i\mathbf{G}^i = D^i\mathbf{G}_i \quad , \quad D_i = \mathbf{G}_i \cdot \mathbf{D} \quad , \quad D^i = G^{ij}D_i \\ \mathbf{D}_{,i} = D^j{}_{|i}\mathbf{G}_j = \Lambda^j{}_i\mathbf{G}_j = \Lambda_{ji}\mathbf{G}^j \end{gathered} \tag{4.39}$$

where we have

$$\Lambda_{ij} = \mathbf{G}_i \cdot \mathbf{D}_{,j} = D_{i\,|\,j} \quad , \quad \Lambda^i{}_j = \mathbf{G}^i \cdot \mathbf{D}_{,j} = D^i{}_{|\,j} \quad , \quad \Lambda^i{}_j = G^{ik}\Lambda_{kj} \tag{4.40}$$

We now introduce relative kinematical measures γ_{ij}, $\mathcal{K}_{ij}$ and γ_i such that

$$\gamma_{ij} = \frac{1}{2}\,(g_{ij} - G_{ij}) = \frac{1}{2}\,(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) = \gamma_{ji} \tag{4.41}$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij} \tag{4.42}$$

and

$$\gamma_i = d_i - D_i \tag{4.43}$$

Making use of (4.7), (4.14), (4.18), (4.32), and (4.39) we may obtain

$$\begin{gathered} \gamma^*_{\alpha\beta} = \gamma^*_{\beta\alpha} = \gamma_{\alpha\beta} = \frac{1}{2}\,\xi(\mathcal{K}_{\alpha\beta} + \mathcal{K}_{\beta\alpha}) + \frac{1}{2}\,\xi^2(\lambda^i{}_\alpha\lambda_{i\beta} - \Lambda^i{}_\alpha\Lambda_{i\beta}) \\ \gamma^*_{\alpha 3} = \gamma^*_{3\alpha} = \frac{1}{2}\,\{\gamma_\alpha + \xi(d^i\lambda_{i\alpha} - D^i\Lambda_{i\alpha})\} \\ \gamma^*_{33} = \frac{1}{2}\,(\mathbf{d} \cdot \mathbf{d} - \mathbf{D} \cdot \mathbf{D}) = \frac{1}{2}\,(d^id_i - D^iD_i) \end{gathered} \tag{4.44}$$

5. Superposed rigid body motion

We recall that when the motion of the body $\mathcal{B}^*$ differs from the given motion by a rigid motion, the position vector $\mathbf{p}^+$ has the form

$$\mathbf{p}^{*+} = \mathbf{p}^{*+}(\eta^i,t') = \mathbf{p}_o^{*+}(t') + Q(t)[\mathbf{p}^*(\eta^i,t) - \mathbf{p}_o^*(t)] \tag{5.1}$$

where Q(t) is a proper orthogonal tensor function of time. Also, under superposed rigid body motions, the position vector $\mathbf{r}$ of the surface s_0 of $\mathcal{B}^*$ changes to

$$\mathbf{r}^+ = \mathbf{r}^+(\theta^i,t') = \mathbf{r}_o^+(t') + Q(t)[\mathbf{r}(\eta^\alpha,t) - \mathbf{r}_o(t)] \tag{5.2}$$

with the help of (5.1) and (5.2) we deduce that the vector function $\mathbf{d}^+(\eta^\alpha,t)$ must transform according to

$$\mathbf{d}^+(\eta^\alpha,t) = Q(t)\mathbf{d}(\eta^\alpha,t) \tag{5.3}$$

under superposed rigid body motion. It is easily seen from (5.3) that the magnitude of $\mathbf{d}(\eta^\alpha,t)$ under superposed rigid body motions remains unchanged:

$$\mathbf{d}^+ \cdot \mathbf{d}^+ = (Q\mathbf{d}) \cdot (Q \cdot \mathbf{d}) = Q^TQ\mathbf{d} \cdot \mathbf{d} = \mathbf{d} \cdot \mathbf{d} \tag{5.4}$$

since for a proper orthogonal tensor Q we have

$$QQ^T = Q^TQ = I \quad , \quad \det(Q) = 1 \tag{5.5}$$

6. Stress-resultants, stress-couples and other definitions for micro-structure

Consider a shell-like three-dimensional micro-structure $\mathcal{B}^*$ bounded by a closed surface $\partial\mathcal{B}^*$, as specified in section 3.2, which consists of the material surfaces (3.6) and a normal (lateral) material surface of the form (3.7) such that ξ = const. are closed smooth curves on the surface (3.7). Let s_1 be a material surface of the form (3.8) lying entirely between s_0 and s_2. Moreover, let $\mathcal{B}^*$ be composed of two shell-like bodies $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ with their lateral surfaces $\partial\mathcal{B}_1^*$ and $\partial\mathcal{B}_2^*$, respectively, as specified in section 11.

Consider an arbitrary part of the material surface $s_0 : \xi = 0$ in the present configuration and let it be denoted by $\hat{\mathcal{P}}$. Also, let $\mathcal{P}^*$, with boundary $\partial\mathcal{P}^*$, refer to an arbitrary part of the shell-like body $\mathcal{B}^*$ in the present configuration such that:

a) $\mathcal{P}^*$ contains $\hat{\mathcal{P}}$;

b) $\partial\mathcal{P}^*$ consists of portions of the surfaces $(3.6)_{1,2}$ and a surface of the form (3.7) at time t;

c) $\partial\mathcal{P}^*$ coincides with $\partial\hat{\mathcal{P}}$ on the surface $s_0 : \xi = 0$.

Moreover, let $\partial\mathcal{P}_l^*$ refer to the part of $\partial\mathcal{P}^*$ specified by a lateral surface of the form (3.7) such that

$$\partial\mathcal{P}_l^* = \partial\mathcal{P}^* = \partial\hat{\mathcal{P}} \quad \text{on} \quad s_0 : \xi = 0 \tag{6.1}$$

Since $\mathcal{B}^*$ is composed of two shell-like bodies $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ separated by the surface $s_1 : \xi = \xi_1$, the part $\mathcal{P}^*$ is also composed of two parts $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ with their corresponding boundaries $\partial\mathcal{P}_1^*$ and $\partial\mathcal{P}_2^*$, respectively.

Let the boundary $\partial\hat{\mathcal{P}}$ of $\hat{\mathcal{P}}$, in the present configuration be denoted by a closed curve c and defined by the position vector $\mathbf{r}$ on $\partial\hat{\mathcal{P}}$. Let

$$\eta^\alpha = \eta^\alpha(s) \tag{6.2}$$

be the parametric equations of the curve c, with s as the arc parameter. Further, let λ and ν denote the unit tangent vector and the outward unit normal to c lying in the surface $s_o : \xi = 0$. Then we have

$$\lambda = \frac{\partial \mathbf{r}(\eta^\alpha(s))}{\partial s} = \lambda^\alpha \, \mathbf{a}_\alpha \ , \ \lambda^\alpha = \frac{d\eta^\alpha(s)}{ds} \tag{6.3}$$

$$\nu = \lambda \times \mathbf{a}_3 = \nu^\alpha \mathbf{a}_\alpha = \nu_\alpha \mathbf{a}^\alpha = \varepsilon^{\alpha\beta}\nu_\alpha \mathbf{a}_\beta \tag{6.4}$$

$$\lambda = \mathbf{a}_3 \times \nu = \mathbf{a}_3 \times \nu_\alpha \mathbf{a}^\alpha = \varepsilon^{\alpha\beta}\nu_\alpha \mathbf{a}_\beta \tag{6.5}$$

where $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ are the ε-symbols in two-dimensional space;

$$\begin{gathered}
\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta 3} = a^{1/2} e_{\alpha\beta} \ , \ \varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta 3} = a^{-1/2} e^{\alpha\beta} \\
e_{11} = e_{22} = 0 \ , \ e^{11} = e^{22} = 0 \\
e_{12} = -\, e_{21} = 1 \ , \ e^{12} = -\, e^{21} = 1
\end{gathered} \tag{6.6}$$

We also recall that the elements of area on the surfaces

$$\begin{gathered}
s_o : \xi = 0 \\
s_2 : \xi = \xi_2(\eta^\alpha) = \text{constant}
\end{gathered} \tag{6.7}$$

are given by

$$da = (g^* g^{*33})^{1/2} d\eta^1 d\eta^2 \qquad \text{for } \xi_2 = \text{const.} \tag{6.8}$$

Moreover, the element of area on the lateral surface $\partial \mathcal{P}_l^*$ is

$$d\mathit{a} = (n^{*1} d\eta^2 - n^{*2} d\eta^1) g^{*1/2} d\xi \tag{6.9}$$

where n_i^* are the components of the outward unit normal to the lateral surface.

Let $\mathbf{N} = \mathbf{N}(\eta^\alpha,t;\nu)$ and $\mathbf{M} = \mathbf{M}(\eta^\alpha,t;\nu)$ be, respectively, the *resultant force* and *resultant couple* vectors, each per unit length of a curve c in the present configuration. We define these resultants as follows:

$$\int_{\partial\hat{\mathcal{P}}} \mathbf{N}\, ds = \int_{\partial\mathcal{P}_1^*} \mathbf{t}^*\, da \quad , \quad \int_{\partial\hat{\mathcal{P}}} \mathbf{M}\, ds = \int_{\partial\mathcal{P}_1^*} \mathbf{t}^*\, \xi\, da \tag{6.10}$$

We also define additional resultants

$$\mathbf{N}^\alpha\, a^{1/2} = \int_o^{\xi_2} \mathbf{T}^{*\alpha} d\xi = \int_o^{\xi_1} \mathbf{T}^{*\alpha} d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} d\xi \tag{6.11}$$

$$\mathbf{M}^\alpha = \int_o^{\xi_2} \mathbf{T}^{*\alpha}\xi\, d\,\xi = \int_o^{\xi_1} \mathbf{T}^{*\alpha} d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*\alpha} d\xi \tag{6.12}$$

$$\mathbf{m}\, a^{1/2} = \int_o^{\xi_2} \mathbf{T}^{*3}\, d\xi = \int_o^{\xi_1} \mathbf{T}^{*3}\, d\xi + \int_{\xi_1}^{\xi_2} \mathbf{T}^{*3}\, d\xi \tag{6.13}$$

Then it can be shown that

$$\mathbf{N} = \mathbf{N}^\alpha \nu_\alpha \quad , \quad \mathbf{M} = \mathbf{M}^\alpha\, \nu_\alpha \tag{6.14}$$

and

$$\mathbf{N}(\nu) = -\mathbf{N}(-\nu) \quad , \quad \mathbf{M}(\nu) = -\mathbf{M}(-\nu) \tag{6.15}$$

We also define two-dimensional (micro-structural) body forces, i.e.,

$$\hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} = \int_o^{\xi_2} \rho^*\, \mathbf{b}^*\, g^{*1/2}\, d\xi + [\bar{\mathbf{t}}^*\, g^{*1/2}(g^{*33})^{1/2}]_{\xi=\xi_2} + [\bar{\mathbf{t}}^*\, g^{*1/2}(g^{*33})^{1/2}]_{\xi=0} \tag{6.16}$$

$$\hat{\rho}\, \hat{\mathbf{l}}\, a^{1/2} = \int_o^{\xi_2} \rho^*\, \mathbf{b}^*\, g^{*1/2}\, \xi\, d\xi + [\bar{\mathbf{t}}^*(g^* g^{*33})^{1/2}\xi]_{\xi=\xi_2} + [\bar{\mathbf{t}}^*(g^* g^{*33})^{1/2}\xi]_{\xi=0} \tag{6.17}$$

where $\bar{\mathbf{t}}^*$ is the value of $\mathbf{t}$ on the boundary $\partial\mathcal{B}^*$ of $\mathcal{B}^*$ (micro-structure). Making use of (8.16), (6.16) and (6.17), we obtain

$$\hat{\rho}\, \hat{\mathbf{f}}\, a^{1/2} = \int_o^{\xi_2} \rho^*\, \mathbf{b}^*\, g^{*1/2}\, d\xi + [\mathbf{T}^{*3}]_o^{\xi_2} = \int_o^{\xi_2} \rho^*\, \mathbf{b}^*\, g^{*1/2}\, d\xi + [\mathbf{T}^{*3}]_{\xi=\xi_2} - [\mathbf{T}^{*3}]_{\xi=0} \tag{6.18}$$

and

$$\hat{\rho}\,\hat{\mathbf{l}}\,a^{1/2} = \int_{o}^{\xi_2} \rho^* \, \mathbf{b}^* \, g^{*1/2} \, \xi \, d\xi + [\mathbf{T}^{*3}\xi]_{o}^{\xi_2} = \int_{o}^{\xi_2} \rho^* \mathbf{b}^* g^{*1/2} \, \xi \, d\xi + [\mathbf{T}^{*3} \, \xi]_{\xi=\xi_2} - [\mathbf{T}^{*3}\xi]_{\xi=0} \tag{6.19}$$

where in obtaining the above formulae we have assumed ξ_2 in $\xi = \xi_2(\eta^\alpha)$ to be constant. Also in obtaining (6.18) and (6.19) we have used

$$\begin{aligned} \mathbf{n}^* &= (g^{*33})^{-1/2}[0,0,-\mathbf{g}^{*3}] \quad \text{on the surface} \quad \mathcal{s}_o : \xi = 0 \\ \mathbf{n}^* &= (g^{*33})^{-1/2}\,[0,0,+\mathbf{g}^{*3}] \quad \text{on the surface} \quad \mathcal{s}_2 : \xi = \xi_2(\eta^\alpha) \end{aligned} \tag{6.20}$$

for the outward unit normal to the surfaces $\mathcal{s}_o$ and $\mathcal{s}_2$. We also define a specific internal energy for the representative element (microstructure) by

$$\hat{\rho} a^{1/2} \hat{\varepsilon} = \int_{o}^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi = \int_{o}^{\xi_1} \rho^* g^{*1/2} \varepsilon^* d\xi + \int_{\xi_1}^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi \tag{6.21}$$

7. Basic field equations for a shell-like representative element (micro-structure)

We now proceed to derive basic field equations of motion for a shell-like representative element (micro-structure) as defined in subsection 3.2. To this end we make use of the various resultants defined in section 6 and procedures described in [Naghdi,1972][1]. Recall the three-dimensional equations of motion in classical continuum mechanics, namely $(2.22)_{1,2}$ and $(2.24)_1$. The derivation is effected by

i) integration of each term in $(2.22)_{1,2}$ and $(2.24)_1$ with respect to ξ between $\xi = 0$ and $\xi = \xi_2$, and

ii) integration after multiplication by ξ of each term in (2.22) with respect to ξ between $\xi = 0$ and $\xi = \xi_2$.

Following this procedure and making use of (3.17), (3.18), (4.1), (4.5) and relevant definition of section 6, we obtain

$$\begin{aligned}
&a: \quad \dot{\hat{\rho}} + \eta^{\alpha}_{\alpha}\hat{\rho} = 0 \\
&b: \quad \mathbf{N}^{\alpha}{}_{|\alpha} + \hat{\rho}\hat{\mathbf{f}} = \hat{\rho}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \\
&c: \quad \mathbf{M}^{\alpha}{}_{|\alpha} - \mathbf{m} + \hat{\rho}\hat{\mathbf{l}} = \hat{\rho}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \\
&d: \quad \mathbf{a}_{\alpha} \times \mathbf{N}^{\alpha} + \mathbf{d} \times \mathbf{m} + \mathbf{d}_{,\alpha} \times \mathbf{M}^{\alpha} = 0 \\
&e: \quad \hat{\rho}(\dot{\hat{\varepsilon}}) = \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w} = \hat{P}
\end{aligned} \tag{7.1}$$

[1] Here, we do not elaborate on the details of the derivation and only record the results. The detail of derivations parallel that of [Naghdi 1972].

where

$$\hat{P} = \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^{\alpha} \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w} \tag{7.2}$$

is the mechanical power of the micro-structure (representative element). These field equations are in their local forms. The global form of these equations will be derived and discussed in the next section.

Before closing this section we proceed to discuss the continuity of stress at the interface of the matrix and reinforcement. To this end we recall the jump conditions in classical continuum mechanics, i.e., (2.44). In particular we consider $(2.44)_2$ and note that since the surface of discontinuity is a material surface and we have $\mathbf{v}^* = \mathbf{u}^*$. Hence we obtain

$$w_n^* = 0 \tag{7.3}$$

and equation $(2.44)_2$ reduces to

$$[\![\mathbf{t}]\!] = 0 \tag{7.4}$$

This shows that at a material surface of two media the stress vector is continuous. Since this result holds for any material surface of two media, we can conclude that within the shell-like body $\mathcal{B}^*$ and at the surface $s_1 : \xi = \xi_1$, the stress vector is continuous.

To ensure the continuity of displacement across the interface we must require for the director to be continuous across the interface. However, at this point, to keep the formulation general, we do not impose such a condition. This is in anticipation that in some cases (such as delamination, micro-buckling), it may be appropriate to admit jump for director.

8. Conservation laws for a shell-like representative element (micro-structure)

This section is concerned with the derivation of the global field equations (conservation laws) for our shell-like representative element (micro-structure). The derivation is accomplished by integrating the basic field equations, derived in section 7, over an appropriate region of two dimensional space covered by η^1, η^2 coordinates. To this end we consider an arbitrary part $\mathcal{P}$ of the materials surface $s_0 : \xi = 0$ in the present configuration and let $\partial\hat{\mathcal{P}}$ be the boundary (curve) of $\hat{\mathcal{P}}$. The basic field equations (in local form) for the part $\mathcal{P}$ were derived in section 7. We recall the following results:

$$J = (\frac{a}{A})^{1/2}$$

$$\dot{a} = \overline{\det(a_{\alpha\beta})}^{\,\cdot} = \frac{\partial}{\partial a_{\lambda\gamma}} [\det(a_{\alpha\beta}]\dot{a}_{\lambda\gamma} = a a^{\alpha\beta}\dot{a}_{\alpha\beta} = 2a\eta^{\alpha}_{\alpha} \tag{8.1}$$

$$\frac{d}{dt}\int_{\partial\hat{\mathcal{P}}} \phi \, d\hat{a} = \int_{\partial\hat{\mathcal{P}}} (\dot{\phi} + \eta^{\alpha}_{\alpha}\phi) d\hat{a}$$

We now integrate both sides of $(7.1)_a$ with respect to η^1,η^2 and make use of $(8.1)_{2,3}$ to obtain

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} a^{1/2}\hat{\rho} d\hat{a} = 0 \tag{8.2}$$

where $d\hat{a}$ is the element of area of the shell-like micro structure. This is the conservation of mass for an arbitrary part $\hat{\mathcal{P}}$ of our shell-like micro-structure.

Next we consider $(7.1)_b$ and integrate with respect to η^1 and η^2 to obtain

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} \rho(v + y^1 w) d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}\hat{f} d\hat{a} + \int_{\hat{\mathcal{P}}} N ds \tag{8.3}$$

where in obtaining (8.3) we have used (6.14) and Stokes' theorem. Equation (8.3) is the conservation of linear momentum for an arbitrary part of the shell-like micro-structure. Following the

same procedure we consider $(7.1)_c$ and integrate with respect to η^1, η^2 and obtain

$$\int_{\hat{\mathcal{P}}} \{[\overline{\hat{\rho}(y^1\mathbf{v} + y^2\mathbf{w}})^{\cdot}] + \eta^{\beta}_{\alpha}[\hat{\rho}(y^1\mathbf{v} + y^2\mathbf{w})]\} d\hat{a} = \int_{\hat{\mathcal{P}}} (\hat{\rho}\hat{\mathbf{l}} - \mathbf{m}) d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{M} ds \tag{8.4}$$

where we have made use of $(6.14)_2$. Equation (8.4) is the conservation of director momentum for an arbitrary part of the shell-like micro-structure.

We now consider $(7.1)_b$, $(7.1)_c$, $(7.1)_d$ and write

$$\hat{\rho}a^{1/2}[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})] = \mathbf{r} \times (\mathbf{N}^{\alpha}a^{1/2})_{,\alpha} + \hat{\rho}\mathbf{r} \times \hat{\mathbf{f}}a^{1/2}$$

$$\hat{\rho}a^{1/2}\,[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})] = \mathbf{d} \times (\mathbf{M}^{\alpha}a^{1/2})_{,\alpha} - \mathbf{d} \times \mathbf{m}a^{1/2} + \hat{\rho}\mathbf{d} \times \hat{\mathbf{l}}a^{1/2} \tag{8.5}$$

$$0 = a^{1/2}(\mathbf{a}_{\alpha} \times \mathbf{N}^{\alpha}) + a^{1/2}(\mathbf{d} \times \mathbf{m}) + a^{1/2}(\mathbf{d}_{,\alpha} \times \mathbf{M}^{\alpha})$$

Adding $(8.5)_{1,2,3}$, we obtain

$$\hat{\rho}\,a^{1/2}[\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})] + \hat{\rho}\,a^{1/2}[\mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})] =$$

$$(a^{1/2}\mathbf{r} \times \mathbf{N}^{\alpha})_{,\alpha} + \hat{\rho}\,a^{1/2}\mathbf{r} \times \mathbf{f} + (a^{1/2}\mathbf{d} \times \mathbf{M}^{\alpha})_{,\alpha} + \hat{\rho}\,a^{1/2}\mathbf{d} \times \hat{\mathbf{l}} \tag{8.6}$$

Integrating (8.6) with respect to η^1, η^2 and making use of $(8.6)_{1,2}$ $(6.4)_{1,2}$ and Stokes' theorem, we obtain

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\{\mathbf{r} \times (\mathbf{v} \times y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\} d\hat{a} =$$

$$\int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}}) d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{r} \times \mathbf{N} + \mathbf{d} \times \mathbf{M}) d\hat{a} \tag{8.7}$$

This is the conservation of moment of momentum of the shell-like micro-structure.

Finally we consider $(7.1)_b$, $(7.1)_c$ and form their scalar products with $\mathbf{v}$ and $\mathbf{w}$ respectively and add the resulting equations to the product of $(7.1)_e$ with $a^{1/2}$ to obtain

$$\hat{\rho}\, a^{1/2}\, (\dot{\hat{\varepsilon}}) + \hat{\rho}\, a^{1/2}(\mathbf{v}\cdot\dot{\mathbf{v}} + y^1\mathbf{v}\cdot\dot{\mathbf{w}} + y^1\dot{\mathbf{v}}\cdot\mathbf{w} + y^2\mathbf{w}\cdot\dot{\mathbf{w}}) =$$

$$\hat{\rho}\, a^{1/2}(\hat{\mathbf{f}}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\mathbf{w}) +$$

$$(a^{1/2}\mathbf{N}^\alpha\cdot\mathbf{v})_{,\alpha} + (a^{1/2}\mathbf{M}^\alpha\cdot\mathbf{w})_{,\alpha} \tag{8.8}$$

Integrating (8.8) with respect to η^1,η^2, making use of Stokes' theorem, and (8.1), (7.1), we obtain

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}}\hat{\rho}(\hat{\varepsilon} + \hat{\mathcal{K}})d\hat{a} = \int_{\hat{\mathcal{P}}}\hat{\rho}(\hat{\mathbf{f}}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\mathbf{w})d\hat{a} + \int_{\partial\hat{\mathcal{P}}}(\mathbf{N}^\alpha\cdot\mathbf{v} + \mathbf{M}^\alpha\cdot\mathbf{w})ds \tag{8.9}$$

where in obtaining (8.9) we have used the fact that

$$\hat{\mathcal{K}} = \frac{1}{2}\,(\mathbf{v}\cdot\mathbf{v} + 2y^1\mathbf{v}\cdot\mathbf{w} + y^2\mathbf{w}\cdot\mathbf{w}) \tag{8.10}$$

Equation (8.9) is the conservation of energy for the shell-like micro-structure. We also define the momentum corresponding to the velocity $\mathbf{v}$ and the director momentum corresponding to $\mathbf{w}$ by

$$\begin{aligned} \hat{\rho}\,\frac{\partial\hat{\mathcal{K}}}{\partial\mathbf{v}} &= \hat{\rho}(\mathbf{v} + y^1\mathbf{w}) \\ \hat{\rho}\,\frac{\partial\hat{\mathcal{K}}}{\partial\mathbf{w}} &= \hat{\rho}(y^1\mathbf{v} + y^2\mathbf{w}) \end{aligned} \tag{8.11}$$

For convenience, we record below the conservation laws for an arbitrary part $\hat{\mathcal{P}}$ bounded by $\partial\mathcal{P}$ of the micro-structure

$$
\begin{aligned}
&a: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\, d\hat{a} = 0 \\
&b: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{v} + y^1\mathbf{w})\, d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}\, \hat{\mathbf{f}}\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{N}\, ds \\
&c: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(y^1\mathbf{v} + y^2\mathbf{w})\, d\hat{a} = \int_{\hat{\mathcal{P}}} (\hat{\rho}\hat{\mathbf{l}} - \mathbf{m})\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} \mathbf{M}\, ds \\
&d: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}\{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}\, d\hat{a} = \\
&\qquad \int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{r} \times \hat{\mathbf{f}} + \mathbf{d} \times \hat{\mathbf{l}})\, d\hat{a} + \int_{\hat{\mathcal{P}}} (\mathbf{r} \times \mathbf{N} + \mathbf{d} \times \mathbf{M})\, ds \\
&e: \ \frac{d}{dt}\int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\varepsilon} + \hat{\mathcal{K}})\, d\hat{a} = \int_{\hat{\mathcal{P}}} \hat{\rho}(\mathbf{f} \cdot \mathbf{v} + \hat{\mathbf{l}} \cdot \mathbf{w})\, d\hat{a} + \int_{\partial\hat{\mathcal{P}}} (\mathbf{N}^\alpha \cdot \mathbf{v} + \mathbf{M}^\alpha \cdot \mathbf{w})\, ds
\end{aligned}
\tag{8.12}
$$

The first of (8.12) is a mathematical statement of the conservation of mass, the second that of the linear momentum, the third is the conservation of the director momentum, the fourth that of the moment of momentum, and the fifth is the conservation of energy. The various quantities appearing in (8.12) have been defined in the previous sections and in what follows we will make reference to these definitions when the need arises.

9. Conservation laws for composite laminates

In this section we derive various conservation laws of a composite laminate from the corresponding conservation laws of a shell-like micro-structure derived in section 8. We recall that the composite laminate is assumed to consist of infinitely many micro-structures. This assumption is justified by physical considerations since the thickness of each ply is small in comparison with the thickness of the laminate itself.

Consider an arbitrary part $\mathcal{P}$ of the composite laminate in the present configuration and let it be bounded by a closed surface $\partial\mathcal{P}$. In view of the choice of the convected curvilinear coordinates θ^i we note that coordinate θ^3 is, roughly speaking, in the direction of the lamination stack up.

Considering the conservation of mass $(8.12)_a$, we write

$$\frac{d}{dt}\int_{\hat{\mathcal{P}}}\hat{\rho}\, d\hat{a} = 0 \quad \Rightarrow \frac{d}{dt}\int_{\hat{\eta}_1}\int_{\hat{\eta}_2}\hat{\rho}\, a^{1/2} d\eta^1 d\eta^2 = \frac{d}{dt}\int_{\bar{\theta}_1}\int_{\bar{\theta}_2}\hat{\rho}\, a^{1/2} d\theta^1 d\theta^2 = 0 \tag{9.1}$$

in view of $(3.3)_1$, where $\bar{\theta}_1, \bar{\theta}_2$ are appropriate ranges of integration within the region $\mathcal{P}$ of the composite laminate. We integrate both sides of (9.1) with respect to θ^3 to obtain

$$\int_{\bar{\theta}_3}\{\frac{d}{dt}\int_{\bar{\theta}_1}\int_{\bar{\theta}_2}\hat{\rho}\, a^{1/2}\, d\theta^1 d\theta^2\} d\theta^3 = \text{const.} \tag{9.2}$$

where $\bar{\theta}_3$ is the appropriate range of integration within the region $\mathcal{P}$. Since coordinates θ^i are convected, and since the quantity $\hat{\rho}a^{1/2}$ is independent of time, we obtain

$$\frac{d}{dt}\int_{\bar{\theta}_3}\int_{\bar{\theta}_2}\int_{\bar{\theta}_1}\hat{\rho}\, a^{1/2}\, d\theta^1 d\theta^2 d\theta^3 = 0 \tag{9.3}$$

The composite element of volume is

$$dv = g^{1/2}\, d\theta^1 d\theta^2 d\theta^3 \tag{9.4}$$

where g is the determinant of the metric of the space covered by the coordinates $\theta^1, \theta^2, \theta^3$. We define *composite assigned mass density*, ρ, such that

$$\rho g^{1/2} = \hat{\rho} a^{1/2} = \int_0^{\xi_2} \rho^* g^{*1/2} d\xi = \int_0^{\xi_1} \rho_1^* g^{*1/2} d\xi + \int_{\xi_1}^{\xi_2} \rho_2^* g^{*1/2}$$

$$\rho = \rho(\theta^i, t) \tag{9.5}$$

where $a = \det(a_{\alpha\beta})$ and $g^* = \det(g^*_{ij})$. Substituting (9.5) into (9.3) and making use of (9.4), we obtain

$$\frac{d}{dt} \int_{\mathcal{P}} g^{1/2} \rho d\upsilon = 0 \tag{9.6}$$

This is the conservation of mass of the composite laminate. From (9.5) it is clear that since $\hat{\rho} a^{1/2}$ is independent of time, it follows that $\rho g^{1/2}$ is also independent of time, although both ρ and $g^{1/2}$ may depend on t. [6]

Next, we consider the conservation of the linear momentum of the micro-structure $(8.12)_b$ and integrate with respect to θ^3 to obtain

$$\int_{\bar{\theta}_3} \{ \frac{d}{dt} \int_{\hat{\mathcal{P}}} \hat{\rho}(v + y^1 w) d\hat{a} \} d\theta^3 = \int_{\bar{\theta}_3} \{ \int_{\hat{\mathcal{P}}} \hat{\rho}\, \hat{f}\, d\hat{a} \} d\theta^3 + \int_{\bar{\theta}_3} \{ \int_{\partial\hat{\mathcal{P}}} N ds \} d\theta^3 + \text{const.} \tag{9.7}$$

We require that in the absence of the body and contact forces the total linear momentum of the composite laminate must remain constant at all times. In view of this and by making use of

6

In this section we will need to perform differentiation with respect to both coordinate systems η^i and θ^i in the same expression. For the sake of clarity in such occasions we will use lower case letters to designate differentiation with respect to η^i coordinates while for the differentiation with respect to θ^i coordinates we will make use of capital letters. For example, $T^{*i}{}_{,i}$ is equal to

$$T^{*i}{}_{,i} = \frac{\partial T^{*1}}{\partial \eta^1} + \frac{\partial T^{*2}}{\partial \eta^2} + \frac{\partial T^{*3}}{\partial \eta^3} = \frac{\partial T^{*1}}{\partial \eta^1} + \frac{\partial T^{*2}}{\partial \eta^2} + \frac{\partial T^{*3}}{\partial \xi}$$

while $T^A{}_{,A}$ is equivalent to

$$T^A{}_{,A} = \frac{\partial T^1}{\partial \theta^1} + \frac{\partial T^2}{\partial \theta^2} + \frac{\partial T^3}{\partial \theta^3}$$

This deviation from our usual notation is temporary and will be adopted when helps to clarify the derivation.

(6.18) and the fact that θ^3 is a convected coordinate we rewrite the above as

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(\mathbf{v}+y^1\mathbf{w})dv=\int_{\mathcal{P}}g^{-1/2}\{\int_o^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}d\xi\}dv+\int_{\mathcal{P}}g^{-1/2}\{\int_o^{\xi_2}\mathbf{T}^{*i}{}_{,i}d\xi\}dv \tag{9.8}$$

where a comma denotes differentiation with respect to $\eta^i=\{\eta^\alpha,\xi\}$. Also, in obtaining (9.8) we have made use of (9.5) and the Stokes' theorem. We now define the *composite assigned body force density*, **b** and the *composite assigned stress vector* $\mathbf{T}^i$ (acting on θ^i = const. surface) such that

$$\rho g^{1/2}\mathbf{b}=\int_o^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}d\xi \tag{9.9}$$

and[7]

$$\mathbf{T}^A{}_{,A}=\mathbf{T}^i{}_{,i}=\int_o^{\xi_2}\mathbf{T}^{*i}{}_{,i}d\xi \tag{9.10}$$

Substituting (9.9) and (9.10) in (9.8) we obtain

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(\mathbf{v}+y^1\mathbf{w})dv=\int_{\mathcal{P}}\rho\mathbf{b}dv+\int_{\mathcal{P}}g^{-1/2}\mathbf{T}^A{}_{,A}dv \tag{9.11}$$

where a comma now denotes differentiation with respect to $\theta^i=\{\theta^1,\theta^2,\theta^3\}$. [8] Making use of the divergence theorem from (9.11) we obtain

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(\mathbf{v}+y^1\mathbf{w})dv=\int_{\mathcal{P}}\rho\mathbf{b}dv+\int_{\partial\mathcal{P}}g^{-1/2}\mathbf{T}^A n_A da \tag{9.12}$$

where $\mathbf{n}=n^i\mathbf{g}_i=n_i\cdot\mathbf{g}^i$ is the outward unit normal to the boundary surface $\partial\mathcal{P}$. Definining[9]

$$\mathbf{t}=g^{-1/2}\mathbf{T}^A n_A \tag{9.13}$$

as the *composite assigned stress vector*, we obtain

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(\mathbf{v}+y^1\mathbf{w})dv=\int_{\mathcal{P}}\rho\mathbf{b}dv+\int_{\partial\mathcal{P}}\mathbf{t}da \tag{9.14}$$

[7,8,9] See footnote #6.

This is the conservation of the linear momentum for the composite laminate.

We now consider the conservation of the director momentum of the micro-structure, $(8.12)_c$ and integrate with respect to θ^3 to obtain

$$\int_{\bar{\theta}_3}\{\frac{d}{dt}\int_{\hat{\mathcal{P}}}\hat{\rho}(y^1\mathbf{v}+y^2\mathbf{w})d\hat{a}\}d\theta^3=\int_{\bar{\theta}_3}\{\int_{\hat{\mathcal{P}}}\hat{\rho}\hat{\mathbf{l}}d\hat{a}\}d\theta^3-\int_{\bar{\theta}_3}\{\int_{\hat{\mathcal{P}}}\mathbf{m}d\hat{a}\}d\theta^3$$

$$+\int_{\bar{\theta}_3}\int_{\partial\hat{\mathcal{P}}}\mathbf{M}ds\,d\theta^3+\text{const.} \tag{9.15}$$

We require that in the absence of body and contact forces the total director momentum of the composite laminate must remain constant at all time. Hence, making use of (6.19) and the fact that θ^3 is a convected coordinate we reduce the above as follows:

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(y^1\mathbf{v}+y^2\mathbf{w})dv=\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}\xi d\xi\}dv-\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}\mathbf{T}^{*3}d\xi\}dv$$

$$+\int_{\mathcal{P}}g^{-1/2}\{\int_0^{\xi_2}(\mathbf{T}^{*i}\xi)_{,i}d\xi\}dv \tag{9.16}$$

where a comma refers to differentiation with respect to $\eta^i=\{\eta^\alpha,\xi\}$. We now define the *composite assigned body couple*, $\mathbf{c}$, the *composite intrinsic director force*, $\mathbf{k}$, and the *composite assigned couple stress vector*, S^i (acting on θ^i = const. surface), respectively, by[10]

$$\rho g^{1/2}\mathbf{c}=\int_0^{\xi_2}\rho^*\mathbf{b}^*g^{*1/2}\xi d\xi \tag{9.17}$$

$$g^{1/2}\mathbf{k}=a^{1/2}\mathbf{m}=\int_0^{\xi_2}\mathbf{T}^{*3}d\xi \tag{9.18}$$

$$S^A{}_{,A}=\int_0^{\xi_2}(\mathbf{T}^{*i}\xi)_{,i}d\xi \tag{9.19}$$

Substituting (9.17) to (9.19) into (9.16), we obtain

[10] See footnote #6.

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(y^1\mathbf{v}+y^2\mathbf{w})dv=\int_{\mathcal{P}}(\rho\mathbf{c}-\mathbf{k})dv+\int_{\mathcal{P}}g^{-1/2}\mathbf{S}^A{}_{,A}dv \tag{9.20}$$

Making use of the divergence theorem, we can reduce (9.20) as follows

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(y^1\mathbf{v}+y^2\mathbf{w})dv=\int_{\mathcal{P}}(\rho\mathbf{c}-\mathbf{k})dv+\int_{\partial\mathcal{P}}g^{-1/2}\mathbf{S}^A n_A da \tag{9.21}$$

Defining

$$\mathbf{s}=g^{-1/2}\mathbf{S}^A n_A \tag{9.22}$$

as the *composite couple traction*, we obtain

$$\frac{d}{dt}\int_{\mathcal{P}}\rho(y^1\mathbf{v}+y^2\mathbf{w})dv=\int_{\mathcal{P}}(\rho\mathbf{c}-\mathbf{k})dv+\int_{\partial\mathcal{P}}\mathbf{s}da \tag{9.23}$$

This is the conservation of the director momentum for the composite laminate.

Next we consider the conservation of moment of momentum for the micro-structure, $(8.11)_d$ and integrate with respect to θ^3

$$\int_{\bar{\theta}_3}\frac{d}{dt}\int_{\hat{\mathcal{P}}}\hat{\rho}\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}d\hat{a}\,d\theta^3=$$

$$\int_{\bar{\theta}_3}\int_{\hat{\mathcal{P}}}\hat{\rho}(\mathbf{r}\times\hat{\mathbf{f}}+\mathbf{d}\times\hat{\mathbf{l}})d\hat{a}d\theta^3+$$

$$\int_{\bar{\theta}_3}\int_{\partial\hat{\mathcal{P}}}(\mathbf{r}\times\mathbf{N}+\mathbf{d}\times\mathbf{M})ds\,d\theta^3+\text{const.} \tag{9.24}$$

We require that in the absence of body and contact forces the angular momentum of the composite laminate must remain constant at all times. In view of this and since θ^3 is a convected coordinate, we may write

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}dv=\int_{\mathcal{P}}\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv$$

$$+\int_{\mathcal{P}} g^{-1/2}(\mathbf{r}\times\int_{o}^{\xi_2}\mathbf{T}^{*i}d\xi)_{,i}dv$$

$$+\int_{\mathcal{P}} g^{-1/2}(\mathbf{d}\times\int_{o}^{\xi_2}\mathbf{T}^{*i}\xi d\xi)_{,i}dv \tag{9.25}$$

where a comma denotes differentiation with respect to $\eta^i=\{\eta^\alpha,\xi\}$. Making use of (9.10) and (9.19) we can rewrite the above as:

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}dv=\int_{\mathcal{P}}\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv$$

$$\int_{\mathcal{P}} g^{-1/2}(\mathbf{r}\times\mathbf{T}^A)_{,A}dv$$

$$\int_{\mathcal{P}} g^{-1/2}(\mathbf{d}\times\mathbf{S}^A)_{,A}dv \tag{9.26}$$

where a comma now refers to differentiation with respect to $\theta^i=\{\theta^1,\theta^2,\theta^3\}$ coordinates. Taking advantage of the divergence theorem, (9.13) and (9.22), we may reduce (9.26) to:

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\{\mathbf{r}\times(\mathbf{v}+y^1\mathbf{w})+\mathbf{d}\times(y^1\mathbf{v}+y^2\mathbf{w})\}dv=$$

$$\int_{\mathcal{P}}\rho(\mathbf{r}\times\mathbf{b}+\mathbf{d}\times\mathbf{c})dv+\int_{\partial\mathcal{P}}(\mathbf{r}\times\mathbf{t}+\mathbf{d}\times\mathbf{s})da \tag{9.27}$$

This is the conservation of the moment of momentum for the composite laminate.

Finally, we consider the conservation of energy for the micro-structure $(8.12)_e$. We recall that in the context of purely mechanical theory $\hat{\varepsilon}=\hat{\varepsilon}(\eta^\alpha,t)$ is the specific internal energy while $\hat{\mathcal{K}}$ represents the kinetic energy of the micro-structure in the present configuration and is given by (8.10). We now integrate both sides of $(8.12)_e$ with respect to θ^3 to obtain

$$\int_{\bar{\theta}_3} \frac{d}{dt} \int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\varepsilon} + \hat{\mathcal{K}}) d\hat{a} d\theta^3 = \int_{\bar{\theta}_3} \int_{\hat{\mathcal{P}}} \hat{\rho}(\hat{\mathbf{f}} \cdot \mathbf{v} + \hat{\mathbf{l}} \cdot \mathbf{w}) d\hat{a} d\theta^3 +$$

$$\int_{\bar{\theta}_3} \int_{\partial \hat{\mathcal{P}}} (\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \mathbf{w}) d\hat{s} d\theta^3 + \text{const.} \tag{9.28}$$

We now require that within the context of purely mechanical theory and in the absence of body and contact forces the total energy of the composite laminate must remain constant at all times. In view of this and since θ^3 is a convected coordinate, we may write

$$\text{Left hand side of (9.28)} = \frac{d}{dt} \int_{\bar{\theta}_3} \int_{\bar{\theta}_2} \int_{\bar{\theta}_1} \left(\int_0^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi + \rho g^{1/2} \hat{\mathcal{K}} \right) d\theta^1 d\theta^2 d\theta^3 \tag{9.29}$$

where in obtaining (9.29) we have made use of (7.1) and (9.5). We now define the *composite assigned strain energy*, and the *composite assigned kinetic* energy, $\mathcal{K}$ both per unit mass of the composite such that

$$\rho g^{1/2} \varepsilon = \int_0^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi \tag{9.29}$$

$$\mathcal{K} = \hat{\mathcal{K}} = \frac{1}{2} (\mathbf{v} \cdot \mathbf{v} + 2y^1 \mathbf{v} \cdot \mathbf{w} + y^2 \mathbf{w} \cdot \mathbf{w}) \tag{9.30}$$

We also record the momentum corresponding to the director velocity $\mathbf{w}$

$$\begin{aligned} \rho \frac{\partial \mathcal{K}}{\partial \mathbf{v}} &= \rho(\mathbf{v} + y^1 \mathbf{w}) \\ \rho \frac{\partial \mathcal{K}}{\partial \mathbf{w}} &= \rho(y^1 \mathbf{v} + y^2 \mathbf{w}) \end{aligned} \tag{9.31}$$

Substituting (9.29) and (9.30) into (9.29), we obtain

$$\text{Left hand side of (9.28)} = \frac{d}{dt} \int_{\mathcal{P}} \rho(\varepsilon + \mathcal{K}) dv \tag{9.32}$$

Considering the right-hand side of (9.28), and making use of (9.9), (9.10), (9.19) and the divergence theorem, we obtain

$$\text{Right hand side of (9.28)} = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w})dv + \int_{\mathcal{P}} \mathbf{t} \cdot \mathbf{v}da + \int_{\mathcal{P}} \mathbf{s} \cdot \mathbf{w}da \tag{9.33}$$

where in obtaining (9.33) we have also made use of (9.13) and (9.22). From (9.32) and (9.33) we obtain

$$\frac{d}{dt} \int_{\mathcal{P}} \rho(\varepsilon + \mathcal{K})dv = dv = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w})dv + \int_{\partial\mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + s \cdot \mathbf{w})da \tag{9.34}$$

This is the composite conservation of energy in purely mechanical theory.

10. Summary of basic principles for composite laminates

Considering the development in the previous section, we are now in a position to state the conservation laws (principles) for composite laminates. In this section we record the conservation laws for composite laminates and also collect various concepts/definitions that were introduced for composite laminates. With reference to the present configuration, these conservation laws are summarized below:

$$
\begin{aligned}
&a: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho \, dv = 0 \\
&b: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{v} + y^1\mathbf{w})dv = \int_{\mathcal{P}} \rho \, \mathbf{b} \, dv + \int_{\partial\mathcal{P}} \mathbf{t} \, da \\
&c: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(y^1\mathbf{v} + y^2\mathbf{w})dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s} \, da \qquad (10.1) \\
&d: \quad \frac{d}{dt}\int_{\mathcal{P}} \{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}dv = \\
&\qquad\qquad \int_{\mathcal{P}} \rho(\mathbf{r} \times \mathbf{b} + \mathbf{d} \times \mathbf{c})dv + \int_{\partial\mathcal{P}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s})da \\
&e: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(\varepsilon + \mathcal{K})dv = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w})dv + \int_{\partial\mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w})da
\end{aligned}
$$

The first of (10.1) is the mathematical statement of conservation of mass, the second that of linear momentum principle, the third that of director momentum, the fourth is the principle of moment of momentum, and the fifth represents the balance of energy for composite laminates.

In (10.1) $\mathbf{r}$, $\mathbf{d}$ denote the *position vector* and the *director* associated with a *composite particle*, respectively, while the velocity and the director velocity of the composite particle are given by $\mathbf{v}$ and $\mathbf{w}$. The definition of the various field quantities in (10.1) and their relation to their counterparts in micro-structure and the similar three dimensional quantities are reiterated below.

1) $\rho = \rho(\theta^i, t)$ is the *composite assigned mass density* in the present configuration given by

$$\rho g^{1/2} = \hat{\rho}\, a^{1/2} = \int_{o}^{\xi_2} \rho^* g^{1/2} d\xi \tag{10.2}$$

where in (10.2) $\hat{\rho}$ is the mass density of the micro-structure, ρ^* is the classical 3-dimensional mass density, g is the determinant of the metric tensor g_{ij} associated with the composite coordinate system θ^i, g^* is the determinant of the metric tensor g^*_{ij} associated with the micro-structure coordinate system $\eta^i = \{\eta^\alpha,\xi\} = \{\theta^\alpha,\xi\}$, a is the determinant of the two-dimensional (surface) metric tensor $a_{\alpha\beta}$ associated with the Cosserat surface (micro-structure).

We notice that the dimensions of ρ^* and $\hat{\rho}$ are mass per unit volume and mass per unit area, respectively. However, the dimension of ρ is the dimension of integrated mass per unit volume of the composite.

2) $\mathbf{b} = \mathbf{b}(\theta^i,t)$ is the *composite assigned body force density* per unit of ρ, given by

$$\rho\, g^{1/2}\, \mathbf{b} = \int_{o}^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* d\xi \tag{10.3}$$

where $\mathbf{b}^*$ is the classical 3-dimensional body force density. The dimension of $\mathbf{b}$ should be clear from (10.3).

3) $\mathbf{c} = \mathbf{c}(\theta^i,t)$ is the *composite assigned body couple density* per unit of ρ, given by[11]

$$\rho\, g^{1/2}\, \mathbf{c} = \int_{o}^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* \xi d\xi \tag{10.4}$$

The dimension of $\mathbf{c}$ is clear from (10.4).

4) $\mathbf{t} = \mathbf{t}(\theta^i,t;\mathbf{n})$ is the *composite assigned stress vector* (per unit area of the composite) such that

$$\mathbf{t} = g^{-1/2}\, \mathbf{T}^i n_i \tag{10.5}$$

[11] $\mathbf{c}$ may also be called "composite assigned director force" emphasizing the "directed" nature of the present continuum theory. In the present context, however, we prefer the terminology in 3 above as it makes the physical nature of $\mathbf{c}$ more apparent.

$$T^A{}_{,A} = T^i{}_{,i} = \int_0^{\xi_2} T^{*i}{}_{,i} d\xi \tag{10.6}$$

$$T^\alpha = \int_0^{\xi_2} T^{*\alpha} d\xi = a^{1/2} N^\alpha \tag{10.7}$$

$$T^3{}_{,3} = T^{*3}{}_{|\xi=\xi_2} - T^{*3}{}_{|\xi=0} = \Delta T^{*3} \tag{10.8}$$

where $\mathbf{T}^{*i}$ is the classical stress vector and $\mathbf{N}^\alpha$ is the resultant force of the micro-structure (i.e., Cosserat surface). We also recall that a comma on the left-hand side of (10.6) to (10.8) denotes partial differentiation with respect to θ^i. However, a comma on the right-hand side of (10.6) and in (10.8) denotes partial differentiation with respect to $\eta^i = \{\eta^\alpha, \xi\}$.

5) $\mathbf{s} = \mathbf{s}(\theta^i, t; \mathbf{n})$ is the *composite assigned couple stress vector* [12] per unit area of the composite such that

$$\mathbf{s} = g^{-1/2} \mathbf{S}^i n_i \tag{10.9}$$

$$\mathbf{S}^i{}_{,i} = \int_0^{\xi_2} \mathbf{T}^{*i}{}_{,i} \xi d\xi \tag{10.10}$$

$$\mathbf{S}^\alpha = \int_0^{\xi} \mathbf{T}^{*\alpha} \xi d\xi = a^{1/2} \mathbf{M}^\alpha \tag{10.11}$$

$$\mathbf{S}^3{}_{,3} = (\mathbf{T}^{*3}\xi)_{|\xi=\xi_2} - (\mathbf{T}^{*3}\xi)_{|\xi=0} = \Delta(\mathbf{T}^{*3}\xi) \tag{10.12}$$

where $\mathbf{M}^\alpha$ is the resultant couple of the micro-structure (i.e., Cosserat surface) and the same remark as in (4) above holds for commas and partial differentiation.

6) $\mathbf{k} = \mathbf{k}(\theta^i, t)$ is the *composite assigned intrinsic (director) force*, per unit volume of the composite, given by

$$g^{1/2} \mathbf{k} = a^{1/2} \mathbf{m} = \int_0^{\xi_2} \mathbf{T}^{*3} d\xi \tag{10.13}$$

[12] $\mathbf{s}$ may also be called "composite assigned contact director force" which reflects the "directed" nature of the present theory. However, the terminology given in 5 reflects the physical nature of $\mathbf{s}$ more clearly.

where $\mathbf{m}$ is the intrinsic director force of the micro-structure (i.e., Cosserat surface).

7) $y^\alpha = y^\alpha(\theta^i)$ are the inertia coefficients which are independent of time and are given by

$$y^\alpha = \int_0^{\xi_2} \rho^* g^{*1/2} \xi^\alpha d\xi \tag{10.14}$$

8) $\varepsilon = \varepsilon(\theta^i,t)$ is the *composite assigned specific internal energy* per unit of ρ given by

$$\rho\, g^{1/2}\, \varepsilon = \hat{\rho}\, a^{1/2}\, \hat{\varepsilon} = \int_0^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi \tag{10.15}$$

where ε^* is the classical 3-dimensional specific internal energy and $\hat{\varepsilon}$ is the specific internal energy per unit $\hat{\rho}$ for the micro-structure (i.e., Cosserat surface).

9) $\mathcal{K} = \mathcal{K}(\theta^i,t)$ is the *composite assigned kinetic energy density* per unit of ρ and is given by

$$\mathcal{K} = \hat{\mathcal{K}} = \frac{1}{2}\,(\mathbf{v}\cdot\mathbf{v} + 2y^1\mathbf{v}\cdot\mathbf{w} + y^2\mathbf{w}\cdot\mathbf{w}) \tag{10.16}$$

where $\hat{\mathcal{K}}$ represents the kinetic energy density per unit $\hat{\rho}$ of the micro-structure (i.e., Cosserat surface). The momentum corresponding to the velocity $\mathbf{v}$ and the director momentum corresponding to $\mathbf{w}$ are given by

$$\rho\,\frac{\partial \mathcal{K}}{\partial \mathbf{v}} = \rho(\mathbf{v} + y^1\mathbf{w}) \tag{10.17}$$

$$\rho\,\frac{\partial \mathcal{K}}{\partial \mathbf{w}} = \rho(y^1\mathbf{v} + y^2\mathbf{w}) \tag{10.18}$$

For simplicity in the rest of this development, when there is no possibility of confusion, we adopt the following simplified terminology:

ρ: "composite mass density"

$\mathbf{b}$: "composite body force density"

$\mathbf{c}$: "composite body couple density"

t: "composite stress vector"

s: "composite couple stress vector"

k: "composite intrinsic force"

ε: "composite specific internal energy"

$\mathcal{K}$: "composite kinetic energy"

We observe that the basic structures of $(10.1)_{a,b,c}$ and their forms are analogous to the corresponding conservation laws of the classical three-dimensional continuum mechanics. Equation $(10.1)_c$ does not exist in the classical continuum mechanics whereas equations $(10.1)_{d,e}$, although they exist, they have simpler forms. It should be noted that the conservation laws (10.1) are consistent with the invariance requirements under superposed rigid body motions, which have wide acceptance in continuum mechanics. We also observe that concepts/quantities such as "body couple density, c_s," "couple stress vector **s**" and "intrinsic force **k**" are not admitted/defined in classical continuum mechanics.

For completeness, we record alternative (simplified) forms of the conservations laws (10.1). Let $\mathcal{P}$ be an arbitrary part (or subset) of the laminated composite body $\mathcal{B}$ with a closed boundary surface $\partial\mathcal{P}$ in the present configuration at time t. The counterparts of $\mathcal{P}$ and $\partial\mathcal{P}$ in a fixed reference configuration will be denoted by $\mathcal{P}_0$ and $\partial\mathcal{P}_0$, respectively. Let ϕ be any scalar or tensor-valued field with the following representation in the present configuration at time t:

$$\phi = \phi(\theta^i, t) \tag{10.19}$$

We recall the transport theorem, i.e.,

$$\int_{\mathcal{P}} (\dot{\phi} + \frac{\dot{g}}{2g}\,\phi) dv = \int_{\mathcal{P}} \frac{\partial \phi}{dt}\, dv + \int_{\partial\mathcal{P}} \phi \mathbf{v} \cdot \mathbf{n}\, da \tag{10.20}$$

With the help of (4.6) and (10.20), conservation laws (10.1) may further be reduced to

$$a: \int_{\mathcal{P}} (\dot{\rho} + \frac{\dot{g}}{2g}\rho) dv = 0$$

$$b: \int_{\mathcal{P}} \rho(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) dv = \int_{\mathcal{P}} \rho \mathbf{b} dv + \int_{\partial\mathcal{P}} \mathbf{t}\, da$$

$$c: \int_{\mathcal{P}} \rho(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) dv = \int_{\mathcal{P}} (\rho\mathbf{c} - \mathbf{k}) dv + \int_{\partial\mathcal{P}} \mathbf{s}\, da \tag{10.21}$$

$$d: \int_{\mathcal{P}} \rho\{\mathbf{r} \times (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) + \mathbf{d} \times (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})\} dv = \int_{\mathcal{P}} \rho(\mathbf{r} \times \mathbf{b} + \mathbf{d} \times \mathbf{c}) dv + \int_{\partial\mathcal{P}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s}) da$$

$$e: \int_{\mathcal{P}} \rho(\dot{\varepsilon} + \dot{\mathcal{K}}) dv = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w}) dv + \int_{\partial\mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w}) da$$

11. Considerations on composite contact force and composite contact couple

Consider an arbitrary part of the composite laminate which occupies a region $\mathcal{P}$ in the present configuration at time t. Let $\mathcal{P}$ be divided into two regions $\mathcal{P}_1,\mathcal{P}_2$ separated by a surface, say σ. Further, let $\partial\mathcal{P},\partial\mathcal{P}_1,\partial\mathcal{P}_2$ refer to the boundaries of $\mathcal{P},\mathcal{P}_1,\mathcal{P}_2$, respectively; and let

$$\partial\overline{\mathcal{P}_1} = \partial\mathcal{P}_1 \cap \partial\mathcal{P} \quad , \quad \partial\overline{\mathcal{P}_2} = \partial\mathcal{P}_2 \cap \partial\mathcal{P} \tag{11.1}$$

Thus, a summary of the above description is as follows:

$$\begin{gathered} \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \quad , \quad \partial\mathcal{P} = \partial\overline{\mathcal{P}_1} \cup \partial\overline{\mathcal{P}_2} \\ \partial\mathcal{P}_1 = \partial\overline{\mathcal{P}_1} \cup \sigma \quad , \quad \partial\mathcal{P}_2 = \partial\overline{\mathcal{P}_2} \cup \sigma \end{gathered} \tag{11.2}$$

We notice that if **n** is the outward unit normal at a point on σ when σ is a portion of $\partial\mathcal{P}_1$, the outward unit normal at the same point on σ when σ is a portion of $\partial\mathcal{P}_2$ is -**n**. Applying the principle of linear momentum, i.e., $(10.22)_b$, to parts $\mathcal{P}_1,\mathcal{P}_2$ and $\mathcal{P}$ and assuming that the composite stress vector is a continuous function of position and **n**, we obtain

$$\mathbf{t}(\theta^i, t; \mathbf{n}) = -\mathbf{t}(\theta^i, t; -\mathbf{n}) \tag{11.3}$$

According to the result (11.3), **the composite stress vector acting on opposite sides of the same surface at a given point are equal in magnitude and opposite in direction**. This is the counterpart of Cauchy's lemma in the classical theory.

Consider an arbitrary part $\mathcal{P}$ of the composite laminate in the present configuration at time t which occupies the region $\mathcal{R}$ in the space covered by the coordinates θ^i. Consider some interior *macro-particle* P of $\mathcal{P}$ having the position vector **r** and the director **d** (note that the particles of the composite laminate are not like ordinary particles in the sense of classical continuum mechanics). We construct at P a curvilinear tetrahedron, lying entirely within $\mathcal{R}$, and in such a way that the side i (i = 1,2,3) is perpendicular to the coordinate direction θ^i and the inclined plane with outward unit normal **n** falls in the octant where $\theta^1,\theta^2,\theta^3$ are all positive. This means that the

edges of the tetrahedron are formed by the coordinate curves PP_i of length ds_i.

We now recall the principle of linear momentum in the form of $(10.21)_b$ and apply it to the tetrahedron under consideration. Following the same procedure as in classical continuum mechanics, we can show the existence of a second order tensor τ^{ij} such that

$$\mathbf{t}_i(g^{ii})^{1/2} = \tau^{ij}\mathbf{g}_j = \tau^i{}_j\mathbf{g}^j \tag{11.4}$$

and

$$\mathbf{t} = \tau^{ij}n_i\mathbf{g}_j = g^{-1/2}\,\mathbf{T}^i\,n_i \tag{11.5}$$

where τ^{ij} and $\tau^i{}_j$ are contravariant and mixed components of the second order tensor which we call the *composite assigned stress tensor* or simply the *composite stress tensor.* We also notice that we have

$$\mathbf{T}^i = g^{1/2}\,\tau^{ij}\,\mathbf{g}_j \tag{11.6}$$

Next we apply the conservation of director momentum in the form of $(10.21)_c$ to parts $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}$. Assuming that the composite contact couple $\mathbf{s}$ is a continuous function of position and $\mathbf{n}$, it follows that

$$\mathbf{s}(\theta^i, t\,;\mathbf{n}) = -\mathbf{s}(\theta^i, t\,;-\mathbf{n}) \tag{11.7}$$

According to the result (11.7), **the composite couple stress (contact couple) vector acting on opposite sides of the same surface at a given point are equal in magnitude and opposite in direction.**

Applying the principle of director momentum in the form $(10.21)_c$ to the tetrahedron and following the usual procedure, we can show the existence of a second order tensor s^{ij} such that

$$\mathbf{s}_i(g^{ii})^{1/2} = s^{ij}\mathbf{g}_j = s^i{}_j\mathbf{g}^j \tag{11.8}$$

$$\mathbf{s} = s^{ij} n_i \mathbf{g}_j = g^{-1/2} \mathbf{S}^i n_i \tag{11.9}$$

where s^{ij} and $s^{i}{}_{j}$ are contravariant and mixed components of the second order tensor which we call the *composite assigned couple stress tensor* or simply the *composite couple stress tensor*. We also notice that we can now write

$$\mathbf{S}^i = g^{1/2} s^{ij} \mathbf{g}_j \tag{11.10}$$

12. Basic field equations for composite laminates

In this section we derive the basic field equations for a composite laminate from the conservation laws of section (11). To this end we consider an arbtirary part $\mathcal{P}$, with boundary $\partial\mathcal{P}$, of the composite body. Then, following the usual procedure and by making use of the transport theorem and the divergence theorem and results of section 11, we deduce from the composite conservation laws (10.22) the following basic field equations.

$$
\begin{aligned}
&a: \quad \dot{\rho} + \frac{\dot{g}}{2g}\rho = 0 \\
&b: \quad \mathbf{T}^i{}_{,i} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \\
&c: \quad \mathbf{S}^i{}_{,i} + g^{1/2}(\rho\mathbf{c} - \mathbf{k}) = \rho g^{1/2}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \\
&d: \quad \mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,i} \times \mathbf{S}^i + g^{1/2}\mathbf{d} \times \mathbf{k} = 0 \\
&e: \quad \rho g^{1/2}\dot{\varepsilon} = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} = g^{1/2}P
\end{aligned}
\tag{12.1}
$$

where P represents the mechanical power density (per element of volume) of the composite laminate and is given by

$$
g^{1/2}P = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} \tag{12.2}
$$

The basic field equations (12.1) are both simple and elegant in form. In practice, however, we usually work with the components of the various fields. Hence, we now proceed to deduce the basic field equations in tensor components. We introduce the contravariant and covariant components of acceleration (α^i, α_i), director acceleration (β^i, β_i), body force (b^i, b_i), body couple (c^i, c_i), and those of intrinsic force (k^i, k_i) as follows:

$$
\begin{aligned}
&\dot{\mathbf{v}} = \alpha^i\mathbf{g}^i = \alpha_i\mathbf{g}^i \quad , \quad \dot{\mathbf{w}} = \beta^i\mathbf{g}_i = \beta_i\mathbf{g}^i \\
&\mathbf{b} = b^i\mathbf{g}_i = b_i\mathbf{g}^i \quad , \quad \mathbf{c} = c^i\mathbf{g}_i = c_i\mathbf{g}^i \quad , \quad \mathbf{k} = k^i\mathbf{g}_i = k_i\mathbf{g}^i
\end{aligned}
\tag{12.3}
$$

Substituting (12.3) into the various field equations in (12.1) and making use of (11.6) and (11.10) and recalling results from tensor analysis, i.e.,

$$g_{j,i} = \{_i{}^k{}_j\} g_k$$
$$(g^{1/2})_{,i} = \frac{1}{2} g^{-1/2} g_{,i} = \{_m{}^m{}_i\} g^{1/2} \tag{12.4}$$

where $\{_j{}^i{}_k\}$ denotes the Christoffel symbol of the second kind, we can obtain the component froms of the equations (12.1). The basic field equations (12.1) when expressed in component forms will reduce to

$$\begin{aligned}
&a: \quad \dot{\rho} + \frac{\dot{g}}{2g}\rho = 0 \\
&b: \quad \tau^{ij}{}_{\|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j) \\
&c: \quad s^{ij}{}_{\|i} + (\rho c^j - k^j) = \rho(y^1\alpha^j + y^2\beta^j) \\
&d: \quad \varepsilon_{ijn}(\tau^{ij} + d^i{}_{\|m}s^{mj} + d^i k^j) = \varepsilon_{ijn}(\tau^{ij} - s^{mi}\lambda'^j_m - k^i d^j) = 0 \\
&e: \quad \rho\dot{\varepsilon} = \tau^{ij}v_{j\|i} + s^{ij}w_{j\|i} + k^i w_i = P
\end{aligned} \tag{12.5}$$

while the expression for mechanical power density P takes the form

$$\rho\dot{\varepsilon} = \tau^{ij}v_{j\|i} + s^{ij}w_{j\|i} + k^i w_i = P \tag{12.6}$$

With reference to $(12.5)_d$ we observe that the symmetry of the stress tensor is not valid. However, because ε_{ijk} is skew-symmetric with respect to i and j, it follows that the quantities in the parentheses in $(12.5)_d$ must be symmetric with respect to i and j. Hence the quantity

$$\tau'^{ij} = \tau^{ij} = s^{mj}\lambda'^j m - k^i d^j \tag{12.7}$$

is symmetric. We call τ'^{ij} the composite assigned symmetric tensor or simply the composite symmetric tensor.

We notice that in the absence of the director, i.e.,

$$\mathbf{d} = 0 \quad \text{or} \quad d^i = 0$$

the composite symmetric tensor T^{ij} reduces to the classical symmetric tensor. It can be shown that in the absence of the micro-structure and the director the basic field equations (12.5) as well as the expressions for power reduce to those of classical continuum mechanics.

13. An elastic composite laminate

The theory developed in the course of this investigation is exact in the context of nonlinear theory and is based on conservation laws that are independent of those in classical continuum mechanics although their derivation was inspired by and started from the classical theory of continuum mechanics. Due to the material and geometric complexities inherent in composites and due to our rather limited (direct) knowledge of composite materials, the study of composite materials has always been conducted via three-dimensional classical continuum mechanics. In particular, constitutive relations for composites have always been derived from those of the constituents which are assumed to be elastic. It is therefore desirable to relate the various field quantities of the composite to those of its constituents. This has already been accomplished, in part, through the relevant definitions in section 10. To complete this correspondence we need to establish appropriate constitutive relationships for the composite field quantities $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}$ This section is concerned with this task.

Within the scope of the theory developed in previous sections, we discuss the constitutive relations for elastic composite laminates in the presence of finite deformation and in the context of purely mechanical theory.

We recall that a material is defined by a constitutive assumption which characterizes the mechanical behavior of the medium. The constitutive assumption places a restriction on the processes which are admissible in a body — here the composite laminate.

We recall that in the three-dimensional theory of classical (non-polar) continuum mechanics and within the context of purely mechanical theory the constitutive relation for the specific internal energy and the stress tensor of an elastic body can be expressed as follows [13]

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[13] Whenever there is no danger of confusion we designate a function and its value with the same symbol. Moreover, the function ψ^* in (13.1) depends also on the reference values G^*_{ij}, but we have not exhibited this here. The partial derivative of a function with respect to a symmetric tensor such as that in (13.2) is understood to have the symmetric form $\frac{1}{2}\left(\frac{\partial\psi^*}{\partial\gamma^*_{ij}} + \frac{\partial\psi^*}{\partial\gamma^*_{ji}}\right)$.

$$\psi^* = \psi^*(\gamma_{ij}) \tag{13.1}$$

$$\tau^{*ij} = \rho^* \frac{\partial \psi^*}{\partial \gamma^*_{ij}} \tag{13.2}$$

We now proceed to deduce the counterparts of the above results for an elastic composite laminate. To this end, we first recall the expression for γ^*_{ij}, i.e.,

$$\gamma^*_{ij} = \gamma^*_{ji} = \frac{1}{2} (\mathbf{g}_i^* \cdot \mathbf{g}_j^* - \mathbf{G}_i^* \cdot \mathbf{G}_j^*)$$

and then observe the following relations[14]:

$$\frac{\partial \psi^*}{\partial \mathbf{g}^*_k} = \frac{\partial \psi^*}{\partial \gamma^*_{ij}} \frac{\partial \gamma^*_{ij}}{\partial \mathbf{g}^*_k} = \frac{\partial \psi^*}{\partial \gamma^*_{ij}} \{ \frac{\partial}{\partial \mathbf{g}^*_k} [\frac{1}{2} (\mathbf{g}_i^* \cdot \mathbf{g}_j^* - \mathbf{G}_i \cdot \mathbf{G}_j)] \}$$

$$= \frac{1}{2} (\frac{\partial \psi^*}{\partial \gamma^*_{kj}} \mathbf{g}_j^* + \frac{\partial \psi^*}{\partial \gamma^*_{ik}} \mathbf{g}_i^*) = \frac{\partial \psi^*}{\partial \gamma^*_{ki}} \mathbf{g}_i^* \tag{13.3}$$

and

$$\frac{\partial \psi^*}{\partial \mathbf{a}_\alpha} = \frac{\partial \psi^*}{\partial \mathbf{g}^*_k} \frac{\partial \mathbf{g}^*_k}{\partial \mathbf{a}^\alpha} = \frac{\partial \psi^*}{\partial \gamma^*_{ki}} \mathbf{g}_i^* \delta^\alpha{}_k = \frac{\partial \psi^*}{\partial \gamma^*_{\alpha i}} \mathbf{g}_i^* \tag{13.4}$$

since

[14] Operators of the form $\frac{\partial f}{\partial \mathbf{x}}$ where f is a scalar valued function of a vector $\mathbf{x} = x^i \mathbf{g}_i = x_i \mathbf{g}^i$ were defined earlier. The component form of this operator which is in fact the gradient operator (derivative operator) is given by

$$\frac{df}{d\mathbf{x}} = \frac{\partial f}{\partial x_i} \mathbf{g}_i = \frac{\partial f}{\partial x^i} \mathbf{g}^i$$

$$\mathbf{g}^*_\beta = \mathbf{a}_\beta + \xi\, \mathbf{d}_{,\beta} \quad , \quad \mathbf{g}^*_3 = \mathbf{d}$$

$$\frac{\partial \mathbf{g}^*_\beta}{\partial \mathbf{a}_\alpha} = \delta^\alpha{}_\beta \quad , \quad \frac{\partial \mathbf{g}^*_3}{\partial \mathbf{a}_\alpha} = 0 \tag{13.5}$$

$$\frac{\partial \mathbf{g}^*_k}{\partial \mathbf{a}_\alpha} = \delta^\alpha{}_k$$

In addition we observe that

$$\frac{\partial \psi^*}{\partial \mathbf{d}} = \frac{\partial \psi^*}{\partial \mathbf{g}^*_k} \frac{\partial \mathbf{g}^*_k}{\partial \mathbf{d}} = \frac{\partial \psi^*}{\partial \gamma^*_{ki}}\, \mathbf{g}_i{}^* \frac{\partial \mathbf{g}^*_k}{\partial \mathbf{d}} = \frac{\partial \psi^*}{\partial \gamma^*_{ki}}\, \mathbf{g}_i{}^*\, \delta^3{}_k = \frac{\partial \psi^*}{\partial \gamma^*_{3i}}\, \mathbf{g}_i{}^* \tag{13.6}$$

where we have made use of (13.3) and the fact that from (13.5) we obtain

$$\frac{\partial \mathbf{g}^*_\beta}{\partial \mathbf{d}} = 0 \quad , \quad \frac{\partial \mathbf{g}^*_3}{\partial \mathbf{d}} = 1 \tag{13.7}$$

$$\frac{\partial \mathbf{g}^*_k}{\partial \mathbf{d}} = \delta^3{}_k$$

Further we observe

$$\frac{\partial \psi^*}{\partial \mathbf{d}_{,\alpha}} = \frac{\partial \psi^*}{\partial \mathbf{g}^*_k} \frac{\partial \mathbf{g}^*_k}{\partial \mathbf{d}_{,\alpha}} = \frac{\partial \psi^*}{\partial \gamma^*_{ki}}\, \mathbf{g}_i{}^* \frac{\partial \mathbf{g}^*_k}{\partial \mathbf{d}_{,\alpha}} = \frac{\partial \psi^*}{\partial \gamma^*_{ki}}\, \mathbf{g}_i{}^*\, \delta^\alpha{}_k\, \xi = \frac{\partial \psi^*}{\partial \gamma^*_{\alpha i}}\, \mathbf{g}_i{}^*\, \xi \tag{13.8}$$

Since from $(13.5)_1$ we have

$$\frac{\partial \mathbf{g}^*_\beta}{\partial \mathbf{d}_{,\alpha}} = \xi\, \delta^\alpha{}_\beta \quad , \quad \frac{\partial \mathbf{g}^*_3}{\partial \mathbf{d}_{,\alpha}} = 0 \tag{13.9}$$

$$\frac{\partial \mathbf{g}^*_k}{\partial \mathbf{d}_{,\alpha}} = \xi\, \delta^\alpha{}_k$$

and we have also made use of (13.3).

We now consider the constitutive equations for the components $\tau^{*\alpha i}$ in (13.2), i.e.,

$$\tau^{*\alpha j} = \rho^* \frac{\partial \psi^*}{\partial \gamma^*_{\alpha j}}$$

Recalling the formula

$$T^{*\alpha} = g^{*1/2}\tau^{*\alpha j*}g_j^*$$

and

$$T^\alpha = \int_o^{\xi_2} T^{*\alpha}\, d\xi = a^{1/2}N^\alpha$$

we write

$$T^\alpha = \int_o^{\xi_2} T^{*\alpha} d\xi = \int_o^{\xi_2} g^{*1/2}\tau^{*\alpha j}g_j^* d\xi = \int_o^{\xi_2} \rho^* g^{*1/2}\,\frac{\partial\psi^*}{\partial\gamma^*_{\alpha j}}\,g_j^* d\xi$$

$$= \int_o^{\xi_2} \rho^* g^{*1/2}\,\frac{\partial\psi^*}{\partial g^*_\alpha}\,d\xi = \int_o^{\xi_2} \rho^* g^{*1/2}\,\frac{\partial\psi^*}{\partial a_\alpha}\,d\xi$$

$$= \frac{\partial}{\partial a_\alpha}\int_o^{\xi_2} \rho^* g^{*1/2}\psi^* d\xi \tag{13.10}$$

We recall that in terms of θ^i coordinates and in relation to a_α and g_α we have

$$\frac{\partial(\;)}{\partial a_\alpha} = \frac{\partial(\;)}{\partial g_\beta}\,\frac{\partial g_\beta}{\partial a_\alpha} = \frac{\partial(\;)}{\partial g_\beta}\,\delta^\beta{}_\alpha = \frac{\partial(\;)}{\partial g_\alpha} \tag{13.11}$$

Hence, we can write (13.10) as

$$T^\alpha = \frac{\partial}{\partial g_\alpha}\int_o^{\xi_2} \rho^* g^{*1/2}\psi^* d\xi \tag{13.12}$$

Next, recall the formula

$$S^\alpha = \int_o^{\xi_2} T^{*\alpha}\xi\, d\xi = a^{1/2}M^\alpha$$

and write

$$S^{\alpha} = \int_{o}^{\xi_2} T^{*\alpha}\xi \, d\xi = \int_{o}^{\xi_2} g^{*1/2}\tau^{*\alpha j}g_j^{*}\xi \, d\xi = \int_{o}^{\xi_2} \rho^{*}g^{*1/2} \frac{\partial\psi^{*}}{\partial\gamma_{\alpha j}^{*}} g_j^{*} \xi \, d\xi$$

$$= \int_{o}^{\xi_2} \rho^{*}g^{*1/2} \frac{\partial\psi^{*}}{\partial\mathbf{d}_{,\alpha}} d\xi = \frac{\partial}{\partial\mathbf{d}_{,\alpha}} \int_{o}^{\xi_2} \rho^{*}g^{*1/2}\psi^{*} d\xi \tag{13.13}$$

where we have made use of (13.8). Now recall the expression

$$g^{1/2}\mathbf{k} = a^{1/2}\hat{\mathbf{k}} = \int_{o}^{\xi} \mathbf{T}^{*3} d\xi$$

and write

$$g^{1/2}\mathbf{k} = \int_{o}^{\xi_2} g^{*1/2}\tau^{*3j}g_j^{*}d\xi = \int_{o}^{\xi_2} \rho^{*}g^{*1/2} \frac{\partial\psi^{*}}{\partial\gamma_{3j}^{*}} g_j^{*} d\xi$$

$$= \int_{o}^{\xi_2} \rho^{*}g^{*1/2} \frac{\partial\psi^{*}}{\partial\mathbf{d}} d\xi = \frac{\partial}{\partial\mathbf{d}} \int_{o}^{\xi_2} \rho^{*}g^{*1/2}\psi^{*}d\xi \tag{13.14}$$

We notice that interlaminar stress vector $\mathbf{T}^3$ acts as an applied contract force for the microstructure. Hence, in general, the constitutive relation for $\mathbf{T}^3$ should be specified directly. This means, in general, $\mathbf{T}^3$ unlike $\mathbf{T}^{\alpha}$ may not be obtained from the strain energy of the constituents.

Consider now the expression

$$\bar{\psi} = \frac{1}{\rho g^{1/2}} \int_{o}^{\xi_2} \rho^{*}g^{*1/2}\psi^{*}(\mathbf{g}_{\alpha}^{*},\mathbf{d},\mathbf{d}_{,\alpha})d\xi \tag{13.15}$$

where the arguments of ψ^{*} have been defined before. Clearly, in view of kinematical relations of section 4, the function $\bar{\psi}$ can be regarded as a function of the variables $\mathbf{g}_i$, $\mathbf{d}$ and $\mathbf{d}_{,i}$. Therefore, the constitutive equations for composite laminate will be given by

$$\psi = \bar{\psi}(\mathbf{g}_i, \mathbf{d}, \mathbf{d}_{,\alpha}) \tag{13.16}$$

$$\mathbf{T}^{i} = \rho g^{1/2} \frac{\partial\bar{\psi}}{\partial\mathbf{g}_i} \tag{13.17}$$

$$S^i = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{d}_{,i}} \tag{13.18}$$

$$g^{1/2}\mathbf{k} = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{d}} \tag{13.19}$$

where S^3 vanishes identically since $\bar{\psi}$ is not a function of $\mathbf{d}_{,3}$. The above constitutive equations are subject to condition $(12.1)_d$.

For completeness, in the rest of this section we obtain the component forms of (13.17)-(13.19). To this end we recall the formulas

$$\mathbf{r} = r^i\mathbf{g}_i = r_i\mathbf{g}^i \quad , \quad \mathbf{d} = d^i\mathbf{g}_i = d_i\mathbf{g}^i \tag{13.20}$$

and

$$\mathbf{r}_{,i} = r^j{}_{|i}\mathbf{g}_j = r_{j|i}\mathbf{g}^j \quad , \quad \mathbf{d}_{,i} = d^j{}_{|i}\mathbf{g}_j = d_{j|i}\mathbf{g}^j \tag{13.21}$$

It is clear from (13.20) and (13.21) that the function $\bar{\psi}$ may be rewritten as

$$\psi = \bar{\psi}(\mathbf{g}_i,\mathbf{d},\mathbf{d}_{,\alpha}) = \bar{\psi}(\mathbf{r}_{,i},\mathbf{d},\mathbf{d}_{,\alpha}) = \tilde{\psi}(r_{m|i},d_m,d_{m|\alpha}) \tag{13.22}$$

With the help of the expression for $\mathbf{T}^i$, (13.20), (13.21) and the gradient of a scalar valued function of a vector, we write

$$\mathbf{T}^i = g^{1/2}\tau^{ij}\mathbf{g}_j = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{g}_i} = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{r}_{,i}} = \rho g^{1/2} \frac{\partial \tilde{\psi}}{\partial r_{j|i}}\, \mathbf{g}_j \tag{13.23}$$

From (13.23) follows that

$$\tau^{ij} = \rho\, \frac{\partial \tilde{\psi}}{\partial r_{j|i}} \tag{13.24}$$

In a similar manner, with the help of expression for S^i, (13.20) and (13.21) we obtain

$$S^i = g^{1/2}s^{ij}\mathbf{g}_j = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{d}_{,i}} = \rho g^{1/2} \frac{\partial \tilde{\psi}}{\partial d_{j|i}}\, \mathbf{g}_j \tag{13.25}$$

From this we obtain

$$s^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial d_{j|i}} \tag{13.26}$$

Next, we consider **k** and making use of the same procedure we write

$$g^{1/2}\mathbf{k} = g^{1/2}k^i\mathbf{g}_i = \rho g^{1/2} \frac{\partial \tilde{\psi}}{\partial \mathbf{d}} = \rho g^{1/2} \frac{\partial \tilde{\psi}}{\partial d_i} \mathbf{g}_i \tag{13.27}$$

and

$$k^i = \rho \frac{\partial \tilde{\psi}}{\partial d_i} \tag{13.28}$$

Collecting the results (13.24), (13.26) and (13.28), we have

$$\tau^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial r_{j|i}} \tag{13.29}$$

$$s^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial d_{j|i}} \tag{13.30}$$

$$k^i = \rho \frac{\partial \tilde{\psi}}{\partial d_i} \tag{13.31}$$

The constitutive equations (13.29) to (13.31) are now subject to condition $(12.5)_d$.

14. The complete theory

We recapitulate in this section the complete theory of elastic composite laminate in the context of purely mechanical theory.

The initial boundary value problem in the general theory.

The basic field equations of the nonlinear theory consist of the equations of motion and the energy equation given by ( ) and repeated below for convenience:

$$\tau^{ij}{}_{|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j) \tag{14.1}$$

$$s^{ij}{}_{|i} + (\rho c^j - k^j) = \rho(y^1\alpha^j + y^2\beta^j) \tag{14.2}$$

$$\varepsilon_{ijn}(\tau^{ij} + \lambda^i{}_n s^{mj} + d^i k^j) = 0 \tag{14.3}$$

$$\rho\dot{\varepsilon} = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^i w_i = P \tag{14.4}$$

where P is given by

$$P = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^i w_i \tag{14.5}$$

or equivalently by

$$P = \mathcal{T}^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i \tag{14.6}$$

The constitutive equations for an elastic composite laminate are specified by

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_i) \tag{14.7}$$

and

$$\mathcal{T}^{ij} = \rho\,\frac{\partial\psi}{\partial\gamma_{ij}} \tag{14.8}$$

$$s^{ij} = \rho \frac{\partial \psi}{\partial \mathcal{K}_{ij}} \tag{14.9}$$

$$k^i = \rho \frac{\partial \psi}{\partial \gamma_i} \tag{14.10}$$

We recall that (14.9) is subjected to the condition

$$s^{i3} = 0 \tag{14.11}$$

We note that instead of (14.7) to (14.10), any other alternative equivalent expressions derived in section   may be used.

The above field equations and constitutive relations characterize the initial boundary-value problem in the nonlinear theory of an elastic composite laminate.

The nature of the boundary conditions in the present theory may be seen from the rate of work expression for the composite contact force and the composite contact couple, i.e.,

$$\mathcal{R}_c = \int_{\partial \mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w}) da \tag{14.12}$$

The conditions at the boundary surface of the composite laminate at which the surface forces $\bar{\mathbf{t}}$ and the surface couples are prescribed require that

$$\mathbf{t} = \bar{\mathbf{t}} \quad , \quad \mathbf{s} = \bar{\mathbf{s}} \tag{14.13}$$

If we express the surface forces $\bar{\mathbf{t}}$ and the surface couples $\bar{\mathbf{s}}$ in terms of their components, i.e.,

$$\bar{\mathbf{t}} = \bar{t}^i \mathbf{g}_i = \bar{t}_i \mathbf{g}^i \tag{14.14}$$

$$\bar{\mathbf{s}} = \bar{s}^i \mathbf{g}_i = \bar{s}_i \mathbf{g}^i \tag{14.15}$$

and then using (   ) and (   ) the boundary conditions take the following forms:

$$\tau^{ij} n_i = \bar{t}^j \quad , \quad \tau^i{}_j n_i = \bar{t}_j \tag{14.16}$$

$$s^{ij}n_i = \bar{s}^j \quad , \quad s^i{}_j n_i = \bar{s}_j \tag{14.17}$$

To elaborate, we recall that our choice of convected coordinates is such that at a point P with coordinates θ^i (i = 1,2,3) of the composite laminate, the coordinates θ^1, θ^2 are in fact the coordinate curves of the ply passing through the point P. Moreover, the coordinate θ^3 is in the direction of lay-up. This implies that for an arbitrary part $\mathcal{P}$, the boundary surface $\partial\mathcal{P}$ consists of two material surfaces of the form

$$\partial\mathcal{P}_1 \;:\quad \theta^3 = \theta^3(\theta^\alpha) = C_1$$

and (14.18)

$$\partial\mathcal{P}_1 \;:\quad \theta^3 = \theta^3(\theta^\alpha) = C_2$$

and a lateral material surface of the form

$$\partial\mathcal{P}_l \;:\quad f(\theta^\alpha) = 0 \tag{14.19}$$

15. A constrained theory of composite laminates

So far our development of the continuum theory has been general and without any restriction/condition placed on the kinematical variables. Therefore the field equations and the constitutive relations are applicable to any elastic composite laminate. We did not introduce any kinematical constraints previously to keep the theory general enough so that it could be utilized for various physical situations. We now turn to the development of a constrained theory of our continuum model. First we derive a set of constraint equations for the composite laminate. We then proceed to obtain the relevant response functions induced by the constraint. Finally we obtain a set of field equations in terms of the displacement and effected by the presence of the constraints.

We impose the condition that plies of the composite laminate do not separate from or slide over each other at all time during the motion of the composite laminate. This means the displacement vector of the material points throughout the body including at the interface must be single valued. Hence we require

$$\mathbf{r}(\theta^\alpha, \theta^3 + \Delta\theta^3) = \mathbf{r}(\theta^\alpha, \theta^3) + \xi_2\, \mathbf{d}(\theta^\alpha, \theta^3)$$

or

$$\frac{\mathbf{r}(\theta^\alpha, \theta^3 + \Delta\theta^3) - \mathbf{r}(\theta^\alpha, \theta^3)}{\xi_2} = \mathbf{d}(\theta^\alpha, \theta^3) \tag{15.1}$$

In the limit when $\xi_2 \to 0$ and $\mathbf{r}(\theta^\alpha, \theta^3 + \Delta\theta^3) \to \mathbf{r}(\theta^\alpha, \theta^3)$ we obtain

$$\mathbf{g}_3 = \mathbf{r}_{,3} = \mathbf{d} \tag{15.2}$$

where we have made use of the assumption

$$\Delta\theta^3 = \xi_2 \tag{15.3}$$

Expression (15.2) implies the following constraint condition

$$\mathbf{g}^{\alpha} \cdot \mathbf{d} = 0 \qquad (\alpha = 1,2) \tag{15.4}$$

Differentiating (15.4) with respect to time, we obtain

$$\dot{\mathbf{g}}^{\alpha} \cdot \mathbf{d} + \mathbf{g}^{\alpha} \cdot \mathbf{w} = 0 \qquad (\alpha = 1,2) \tag{15.5}$$

We recall the formulas

$$\mathbf{g}^{i} \cdot \mathbf{g}_{j} = \delta^{i}_{j} \tag{15.6}$$

Differentiating (15.6) with respect to time, we obtain

$$\dot{\mathbf{g}}^{i} = -\, v^{i}{}_{|m} \mathbf{g}^{m} \tag{15.7}$$

Substituting (15.7) into (15.5), we arrive at

$$d^{i} \mathbf{g}^{\alpha} \cdot \mathbf{v}_{,i} - \mathbf{g}^{\alpha} \cdot \mathbf{w} = 0 \quad (\alpha = 1,2) \tag{15.8}$$

or

$$d_{m} g^{im} g^{j\alpha} v_{j|i} - g^{i\alpha} w_{i} = 0 \qquad (\alpha = 1,2) \tag{15.9}$$

This is another form of the constraints (15.5) which is more appropriate for our present development.

For a composite laminate with constraints we assume that each of the functions $\mathbf{T}^i$, $\mathbf{S}^i$ and $\mathbf{k}^i$ are determined to within an additive constraint response so that

$$\begin{aligned} \mathbf{T}^{i} &= \tilde{\mathbf{T}}^{i} + \hat{\mathbf{T}}^{i} \\ \mathbf{S}^{i} &= \tilde{\mathbf{S}}^{i} + \hat{\mathbf{S}}^{i} \\ \mathbf{k} &= \tilde{\mathbf{k}} + \hat{\mathbf{k}} \end{aligned} \tag{15.10}$$

where

$$\hat{\mathbf{T}}^i \ , \ \hat{\mathbf{S}}^i \ , \ \hat{\mathbf{k}} \tag{15.11}$$

are specified by constitutive equations and

$$\bar{\mathbf{T}}^i \ , \ \bar{\mathbf{S}}^i \ , \ \bar{\mathbf{k}} \tag{15.12}$$

which represent the response due to constraints (15.8) are arbitrary functions of θ^i,t, are workless and independent of the kinematical variables $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$. Thus, recalling the expression (12.2) for mechanical power, we set

$$\bar{\mathbf{T}}^i \cdot \mathbf{v}_{,i} + \bar{\mathbf{S}}^i \cdot \mathbf{w}_{,i} + g^{1/2}\bar{\mathbf{k}} \cdot \mathbf{w} = 0 \tag{15.13}$$

This must hold for all values of the variables $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$ subject to the constraint condition (15.8). Multiplying (15.8) by the Lagrange multipliers δ_α $(\alpha = 1,2)$ and subtracting the results from (15.13), we obtain

$$(\bar{\mathbf{T}}^i - \delta_\alpha d^i \mathbf{g}^\alpha) \cdot \mathbf{v}_{,i} + \bar{\mathbf{S}}^i \cdot \mathbf{w}_{,i} + (g^{1/2}\bar{\mathbf{k}} + \delta_\alpha \mathbf{g}^\alpha) \cdot \mathbf{w} = 0 \tag{15.14}$$

From (15.14) and the fact that $\bar{\mathbf{T}}^i$, $\bar{\mathbf{S}}^i$ and $\bar{\mathbf{k}}$ are independent of $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$ it follows that

$$\bar{\mathbf{T}}^i = \delta_\alpha d^i \mathbf{g}^\alpha \tag{15.15}$$

$$\bar{\mathbf{S}}^i = 0 \tag{15.16}$$

$$g^{1/2}\bar{\mathbf{k}} = -\,\delta_\alpha \mathbf{g}^\alpha \tag{15.17}$$

Expressions (15.15) to (15.17) represent the constraint response induced by the constraint equations (15.3). Substituting (15.15) to (15.17) into linear momentum equation $(12.1)_b$ and the director momentum equation $(12.1)_c$, we obtain

$$[\hat{\mathbf{T}}^i + \delta_\alpha d^i \mathbf{g}^\alpha]_{,i} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) \tag{15.18}$$

and

$$\hat{S}^i{}_{,i} + \rho g^{1/2}\mathbf{c} - [g^{1/2}\hat{\mathbf{k}} - \delta_\alpha g^\alpha] = \rho g^{1/2}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \tag{15.19}$$

Introducing the following temporary variables $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ by

$$\hat{\mathbf{b}} = \mathbf{b} - (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})$$

and (15.20)

$$\hat{\mathbf{c}} = \mathbf{c} - (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})$$

from (15.18) and (15.19) we obtain

$$\rho g^{1/2}\hat{\mathbf{b}} + \hat{\mathbf{T}}^i{}_{,i} - (\rho g^{1/2} d^i\hat{\mathbf{c}} + d^i\hat{\mathbf{S}}^j{}_{,j} - g^{1/2} d^i\mathbf{k})_{,i} = 0 \tag{15.21}$$

Moreover, from (15.19) and (15.4) we obtain

$$\rho g^{1/2}\mathbf{d}\cdot\hat{\mathbf{c}} + \mathbf{d}\cdot\hat{\mathbf{S}}^j{}_{,j} - g^{1/2}\mathbf{d}\cdot\hat{\mathbf{k}} = 0 \tag{15.22}$$

Recalling that $\hat{\mathbf{T}}^i$, $\hat{\mathbf{S}}^i$ and $\hat{\mathbf{k}}$ are specified as functions of various kinematical variables, it is clear that the system of equations (15.21) and (15.22) represent two equations for the determination of the primary unknowns $\mathbf{v}$ (or $\mathbf{r}$) and $\mathbf{d}$.

To obtain the counterparts of (15.21) and (15.22) in component form the functions τ^{ij}, s^{ij} and k^i are determined to within an additive constraint response so that

$$\tau^{ij} = \tilde{\tau}^{ij} + \hat{\tau}^{ij}$$

$$s^{ij} = \tilde{s}^{ij} + \hat{s}^{ij} \tag{15.23}$$

$$k^i = \tilde{k}^i + \hat{k}^i$$

where

$$\hat{\tau}^{ij} \ , \ \hat{s}^{ij} \ , \ \hat{k}^i \tag{15.24}$$

are specified by constitutive equations and

$$\tilde{\tau}^{ij} \ , \ \tilde{s}^{ij} \ , \ \tilde{k}^i \tag{15.25}$$

which represent the response due to constraints (15.9), are arbitrary functions of θ^i,t, workless and independent of kinematical variables $v_{i|j}$, $w_{i|j}$ and w_i. Thus, recalling the expression (12.6) for mechanical power, we set

$$\tilde{\tau}^{ij}v_{j|i} + \tilde{s}^{ij}w_{j|i} + \tilde{k}^i w_i = 0 \tag{15.26}$$

which must hold for all values of the variables $v_{j|i}$, $w_{j|i}$ and w_i subject to the constraint conditions (15.9). Multiplying (15.9) by the Lagrange multipliers λ_α ($\alpha = 1,2$) and subtracting the results from (15.26), we obtain [15]

$$(\tilde{\tau}^{ij} - \lambda_\alpha d^i g^{j\alpha})v_{j|i} + \tilde{s}^{ij}w_{j|i} + (\tilde{k}^i + \lambda_\alpha g^{i\alpha})w_i = 0 \tag{15.27}$$

From (15.27) and the fact that $\tilde{\tau}^{ij}$, $\tilde{s}^{ij}$ and $\tilde{k}$ are independent of $v_{j|i}$, $w_{j|i}$ and w_i it follows that

$$\tilde{\tau}^{ij} = \lambda_\alpha d^i g^{j\alpha} \tag{15.28}$$

$$\tilde{s}^{ij} = 0 \tag{15.29}$$

$$\tilde{k}^i = -\lambda_\alpha g^{i\alpha} \tag{15.30}$$

Substituting (15.28) to (15.30) into $(12.5)_b$ and $(12.5)_c$, we obtain

$$[\hat{\tau}^{ij} + \lambda_\gamma d^i g^{j\gamma}]_{|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j) \tag{15.31}$$

and

$$\hat{s}^{ij}{}_{|i} - [\hat{k}^j - \lambda_\gamma g^{j\gamma}] + \rho c^j = \rho(y^1\alpha^j + y^1\beta^j) \tag{15.32}$$

Making use of (15.20), from (15.31) and (15.32) we obtain

$$\rho\hat{b}^j + \hat{\tau}^{ij}{}_{|i} - (\rho d^i\hat{c}^j + \rho d^i\hat{s}^{mj}{}_{|m} - d^i\hat{k}^j)_{|i} = 0 \tag{15.33}$$

Moreover, from (15.32) and (15.4) we obtain

---

[15] Note that λ_α is now different from δ_α.

$$\rho d_j \hat{c}^j + d_j \hat{s}^{ij}{}_{|i} - d_j \hat{k}^j = 0 \tag{15.34}$$

Again recalling that $\hat{\tau}^{ij}$, $\hat{s}^{ij}$ and $\hat{k}^i$ are specified, by constitutive equations, as functions of relevant kinematical variables, it is clear that the system of equations (15.33) and (15.34) represent four equations for the determination of four primary unknowns v_i and d.

Before closing this section, we obtain a relation between the Lagrange multipliers δ_α and λ_α. Recalling (11.6), we may write (15.15) as follows

$$\bar{\mathbf{T}}^i = g^{1/2}\tilde{\tau}^{ij}\mathbf{g}_j = \delta_\alpha d^i g^{j\alpha}\mathbf{g}_j$$

or

$$g^{1/2}(\tilde{\tau}^{ij} - g^{-1/2}\delta_\alpha d^i g^{j\alpha})\mathbf{g}_j = 0 \tag{15.35}$$

Since $g^{1/2} \neq 0$ and $\mathbf{g}_j$ are linearly independent base vectors, we obtain

$$\tilde{\tau}^{ij} = g^{-1/2}\delta_\alpha d^i g^{j\alpha} \tag{15.36}$$

A comparison between (15.36) and (15.28) yields

$$\lambda_\alpha = g^{-1/2}\delta_\alpha \tag{15.37}$$

Similarly, from (15.17) we obtain

$$g^{1/2}(\tilde{k}^i + g^{-1/2}\delta_\alpha g^{i\alpha}\mathbf{g}_i) = 0 \tag{15.38}$$

Again, since $\mathbf{g}_i$ are independent base vectors and since $g^{1/2} \neq 0$, we obtain

$$\tilde{k}^i = -\, g^{-1/2}\delta_\alpha g^{i\alpha} \tag{15.39}$$

Comparing (15.39) and (15.30), we obtain

$$\lambda_\alpha = g^{-1/2}\delta_\alpha \tag{15.40}$$

which confirms the results (15.37).

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DERIVATION OF CONSTITUTIVE RELATIONS FOR COMPOSITE LAMINATES

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on

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in

Dayton, Ohio

October 1991

BASE

DERIVATION

OF

CONSTITUTIVE RELATIONS

FOR

COMPOSITE LAMINATES

1. Overview

1.a Objective:

To develop a continuum theory for laminated composite materials such that

i) It accounts for the effects of micro-structure
ii) It accounts for the effects of geometric nonlinearity
iii) It accounts for the effects of material nonlinearity
iv) It accounts for the effects of curvature
v) It accounts for the effects of interlaminar stresses
vi) It has a continuum character
vii) It is applicable to both static and dynamic analysis.

1.b Approach:

The above goals will be accomplished by utilizing the following:

i) Convected curvilinear coordinates.
ii) General tensor analysis
iii) Classical three-dimensional continuum mechanics
iv) Theory of Cosserat (directed) surfaces

2. Basic field equations of classical continuum mechanics in general curvilinear coordinates

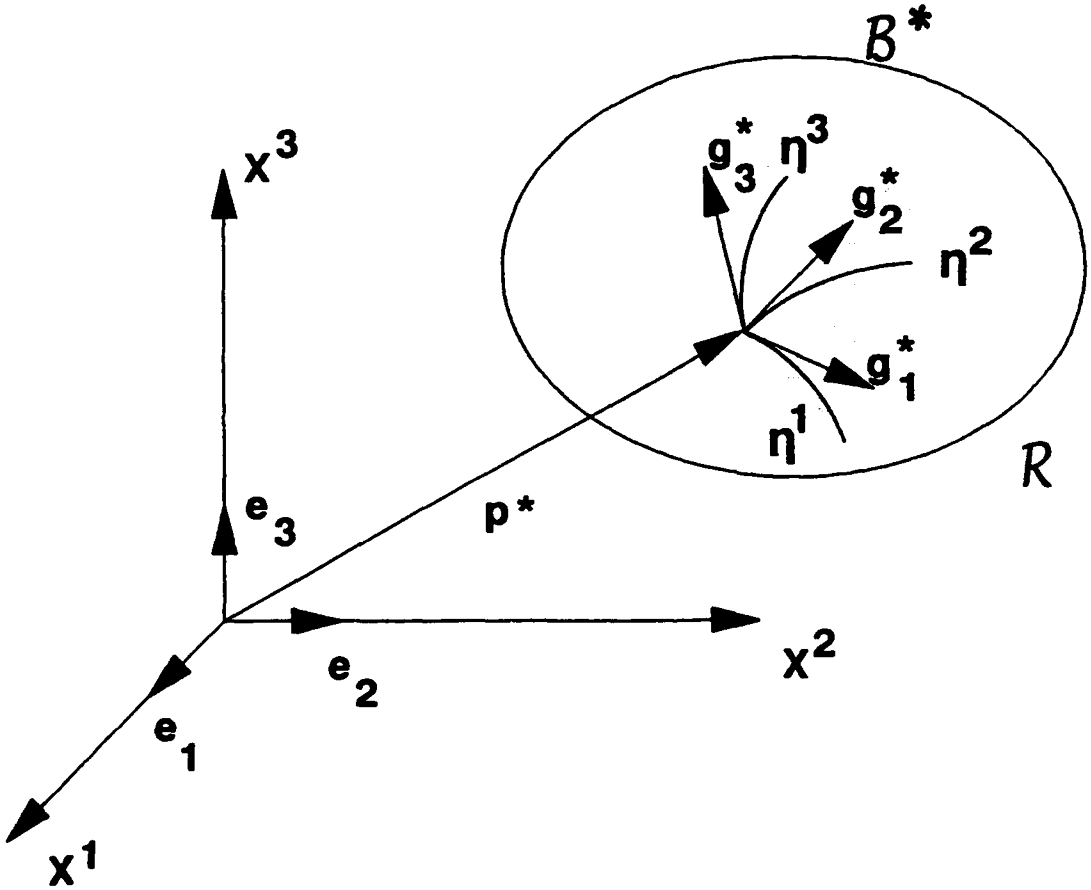

Figure 1
A continuum body in Euclidean 3-space

Let η^i (i = 1,2,3) be a set of general convected curvilinear coordinates. Consider a body $\mathcal{B}^*$ which occupies a region $\mathcal{R}$ in three dimensional Euclidean space and let its boundary be a closed surface and be denoted by $\partial\mathcal{B}^*$. Let

$$\mathbf{p}^* = \mathbf{p}^*(\eta^i,t) \tag{2.1}$$

denote the position vector of a material point P^* in the present configuration of the body $\mathcal{B}^*$ at time t. Then we may write

$$\mathbf{g}_i^* = \frac{\partial \mathbf{p}^*}{\partial \eta^i} \quad , \quad g_{ij}^* = \mathbf{g}_i^* \cdot \mathbf{g}_j^* \tag{2.2}$$

and

$$ds^2 = d\mathbf{p}^* \cdot d\mathbf{p}^* = g_{ij}^* d\eta^i d\eta^j \tag{2.3}$$

where (2.2) and (2.3) are the convariant base vectors, the metric tensor, and the square of a line element in the present configuration at time t, respectively. Similarly in the reference configuration we have

$$\mathbf{P}^* = \mathbf{P}^*(\eta^i) \tag{2.4}$$

$$\mathbf{G}_i^* = \frac{\partial \mathbf{P}^*}{\partial \eta^i} \quad , \quad G_{ij}^* = \mathbf{G}_i^* \cdot \mathbf{G}_j^* \tag{2.5}$$

$$dS^2 = d\mathbf{P}^* \cdot d\mathbf{P}^* = G_{ij}^* d\eta^i d\eta^j \tag{2.6}$$

In addition, we define a strain measure through

$$ds^2 - dS^2 = 2\gamma_{ij}^* d\eta^i d\eta^j \tag{2.7}$$

$$\gamma_{ij}^* = 1/2(g_{ij}^* - G_{ij}^*) \tag{2.8}$$

where γ_{ij}^* are the covariant components of the symmetric strain tensor. Moreover, the velocity is given by

$$\mathbf{v}^* = \dot{\mathbf{p}}^* = \frac{\partial \mathbf{p}^*}{\partial t} \tag{2.9}$$

With reference to the present configuration and within the context of the classical (nonpolar) continuum mechanics, the basic field equations in purely mechanical theory are given by:

$$\begin{aligned}
&\text{a:}\quad \dot{\rho}^* + \frac{\dot{g}^*}{2g^*}\,\rho^* = 0 \\
&\text{b:}\quad \mathbf{T}^i{}_{,i} + \rho^*\mathbf{b}^* g^{*1/2} = \rho^*\dot{\mathbf{v}}^* g^{*1/2} \\
&\text{c:}\quad \mathbf{g}_i \times \mathbf{T}^i = 0 \\
&\text{d:}\quad \rho^* g^{*1/2}\dot{\varepsilon}^* = \mathbf{T}^{*i}\cdot \mathbf{v}^*_{,i}
\end{aligned} \tag{2.10}$$

where we have

$$\begin{aligned}
\mathbf{t}^* &= \frac{\mathbf{T}^{*i} n_i^*}{g^{*1/2}} = \tau^{*ij} n_i^* \mathbf{g}_j^* \quad , \\
\mathbf{T}^{*i} &= g^{*1/2}\tau^{*ij}\mathbf{g}_j^* = g^{*1/2} t_j^{*i}\mathbf{g}^{*j}
\end{aligned} \tag{2.11}$$

3. The basic theory of Cosserat (directed) surfaces

Consider a material surface embedded in a Euclidean 3-space, together with a deformable vector field, called director, attached to every point of the material surface. The director is not necessarily along the unit normal to the surface and remains unaltered in length under superposed rigid body motions. Let the particles of the material surface, say $\mathcal{s}$, be identified by means of a system of convected coordinates η^α (α = 1,2). Let **r** and **d** denote the position vector of a typical point $\hat{P}$ of $\mathcal{s}$ and the director at the same point, respectively.

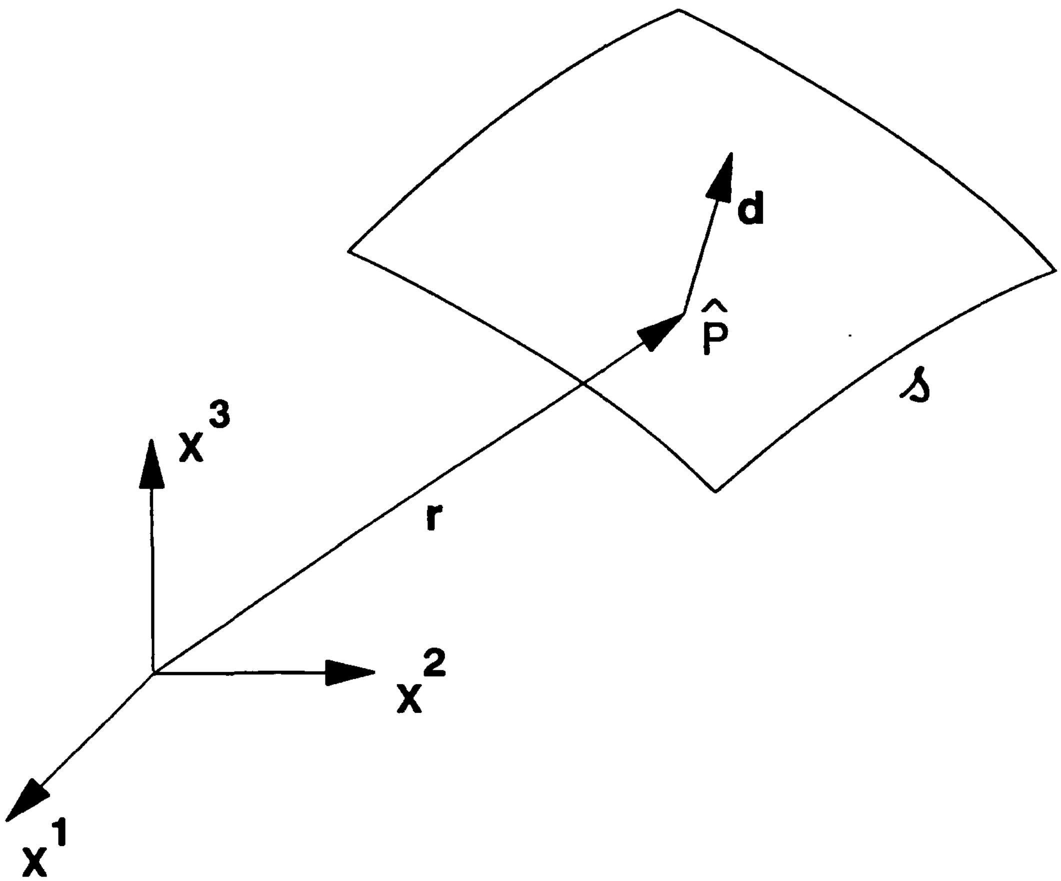

Figure 2
A typical Cosserat surface

Then the motion of the Cosserat surface is defined by vector-valued functions which assign position $\mathbf{r}$ and director $\mathbf{d}$ to each particle $\hat{\mathrm{P}}$ of $\mathcal{s}$ at each instant of time through

$$\mathbf{r} = \mathbf{r}(\eta^{\alpha},t) \quad , \quad \mathbf{d} = \mathbf{d}(\eta^{\alpha},t) \quad , \quad [\mathbf{a}_1\mathbf{a}_2\mathbf{d}] > 0 \tag{3.1}$$

where

$$\mathbf{a}_{\alpha} = \mathbf{a}_{\alpha}(\eta^{\alpha},t) = \frac{\partial \mathbf{r}}{\partial \eta^{\alpha}} \tag{3.2}$$

are the base vectors along the η^{α}-curves on $\mathcal{s}$. The velocity and the director velocity vectors are defined by

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{w} = \dot{\mathbf{d}} \tag{3.3}$$

With reference to the present configuration, the field equations of a Cosserat surface in the context of purely mechanical theory are given by

$$a: \quad \overline{(\hat{\rho} a^{1/2})}^{\,\cdot} = 0$$

$$b: \quad \hat{\rho} a^{1/2}(\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) = (\mathbf{N}^\alpha a^{1/2})_{,\alpha} + \hat{\rho}\hat{\mathbf{f}} a^{1/2}$$

$$c: \quad \hat{\rho} a^{1/2}(y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) = (\mathbf{M}^\alpha a^{1/2})_{,\alpha} - \mathbf{m} a^{1/2} + \hat{\rho}\hat{\mathbf{l}} a^{1/2} \tag{3.4}$$

$$d: \quad \mathbf{a}_\alpha \times \mathbf{N}^\alpha + \mathbf{d} \times \mathbf{m} + \mathbf{d}_{,\alpha} \times \mathbf{M}^\alpha = 0$$

$$e: \quad \hat{\rho}(\dot{\hat{\varepsilon}}) = \mathbf{N}^\alpha \cdot \mathbf{v}_{,\alpha} + \mathbf{M}^\alpha \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w}$$

The first of (3.4) is a mathematical statement of the conservation of mass, the second that of the linear momentum, the third is the conservation of the director momentum, the fourth that of the moment of momentum, and the fifth is the conservation of energy. The various quantities appearing in the above conservation laws are defined below:

$\hat{\rho}$: mass density of the surface s

$\mathbf{N} = \mathbf{N}(\eta^\alpha, t) = \mathbf{N}^\alpha(\eta^\alpha, t)\nu_\alpha$: the contact force

$\mathbf{M} = \mathbf{M}(\eta^\alpha, t) = \mathbf{M}^\alpha(\eta^\alpha, t)\nu_\alpha$: the contact director force

where $\nu_\alpha = \nu_\alpha(\eta^\alpha, t)$ are the components of the outward unit normal to the boundary of the shell-like body

$\hat{\mathbf{f}} = \hat{\mathbf{f}}(\eta^\alpha, t)$: the assigned force

$\hat{\mathbf{l}} = \hat{\mathbf{l}}(\eta^\alpha, t)$: the assigned director force

$\mathbf{m} = \mathbf{m}(\eta^\alpha, t)$: the intrinsic director force

$a = a(\eta^\alpha, t)$: determinant of the first fundamental form of the surface

$y^\alpha = y^\alpha(\eta^\alpha)$: the inertia coefficients

$\hat{\varepsilon} = \hat{\varepsilon}(\eta^\alpha, t)$: the specific internal energy

$\hat{\mathcal{K}} = \hat{\mathcal{K}}(\eta^\alpha, t)$: the kinetic energy of the surface s

The above quantities are related to the corresponding three-dimensional quantities of the shell-like body and are obtained through suitable integration procedures.

4. Modeling of a composite laminate as a series of Cosserat (directed) surfaces

We define a composite laminate as a three-dimensional continuum consisting of multiple layers (two or more) of materials which act together as a single (integral) physical entity. Here we confine our attention to laminated composites composed of multiple layers of only two materials, each of which are considered to be homogeneous. The layers are **not** considered to be necessarily **flat** and could have any type of curvature.

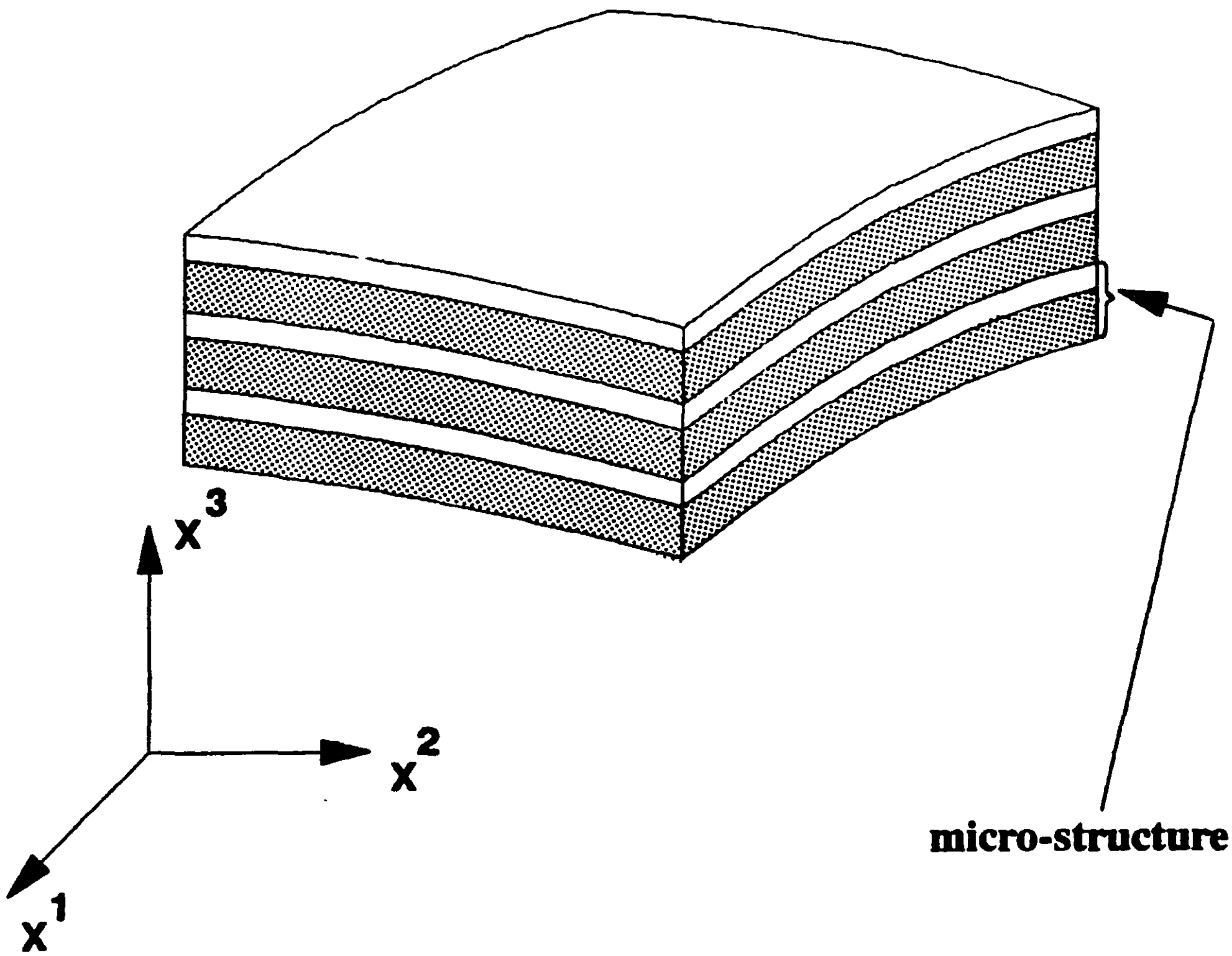

Figure 3
A composite laminate consisting of alternating layers of two materials

Step 1. Selection of a micro-structure:

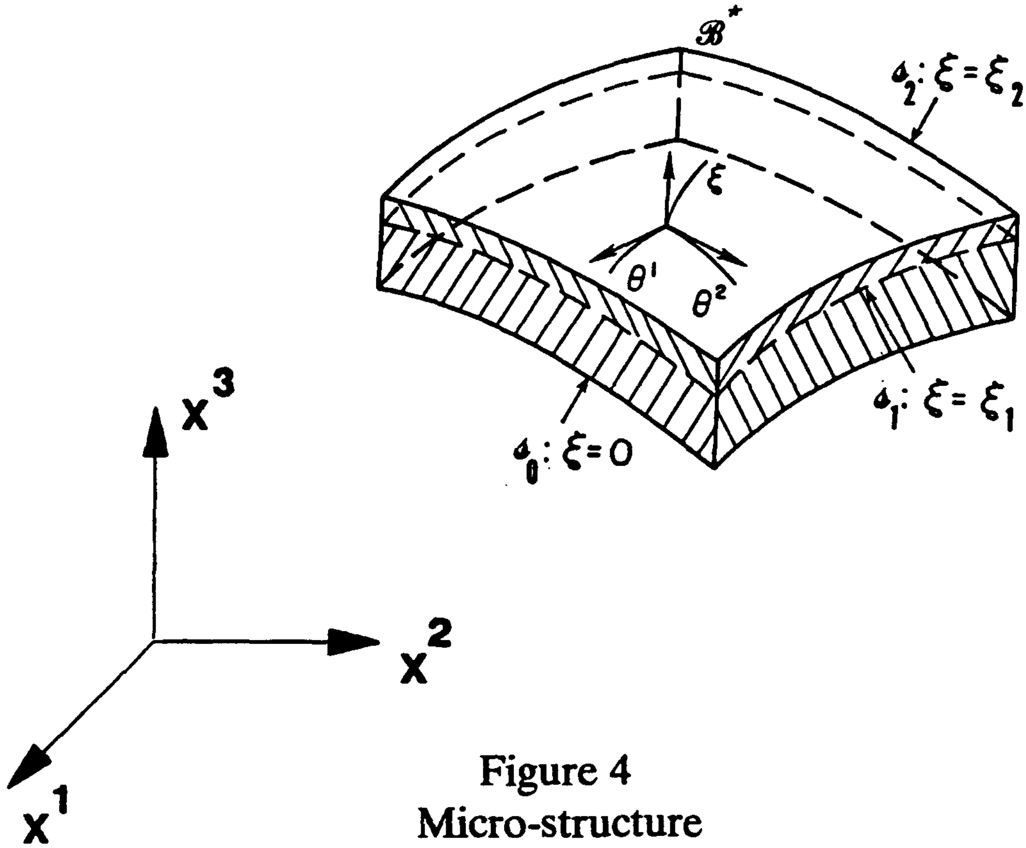

Figure 4
Micro-structure

Step 2. Continuum modeling of the micro-structure

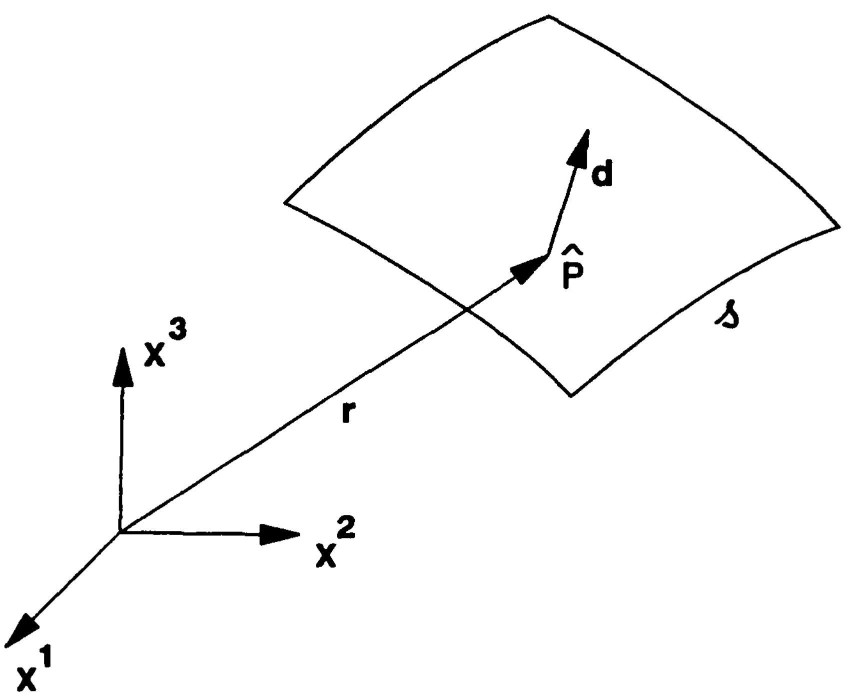

Figure 5
Cosserat surface

Step 3. Continuum modeling of composite laminate

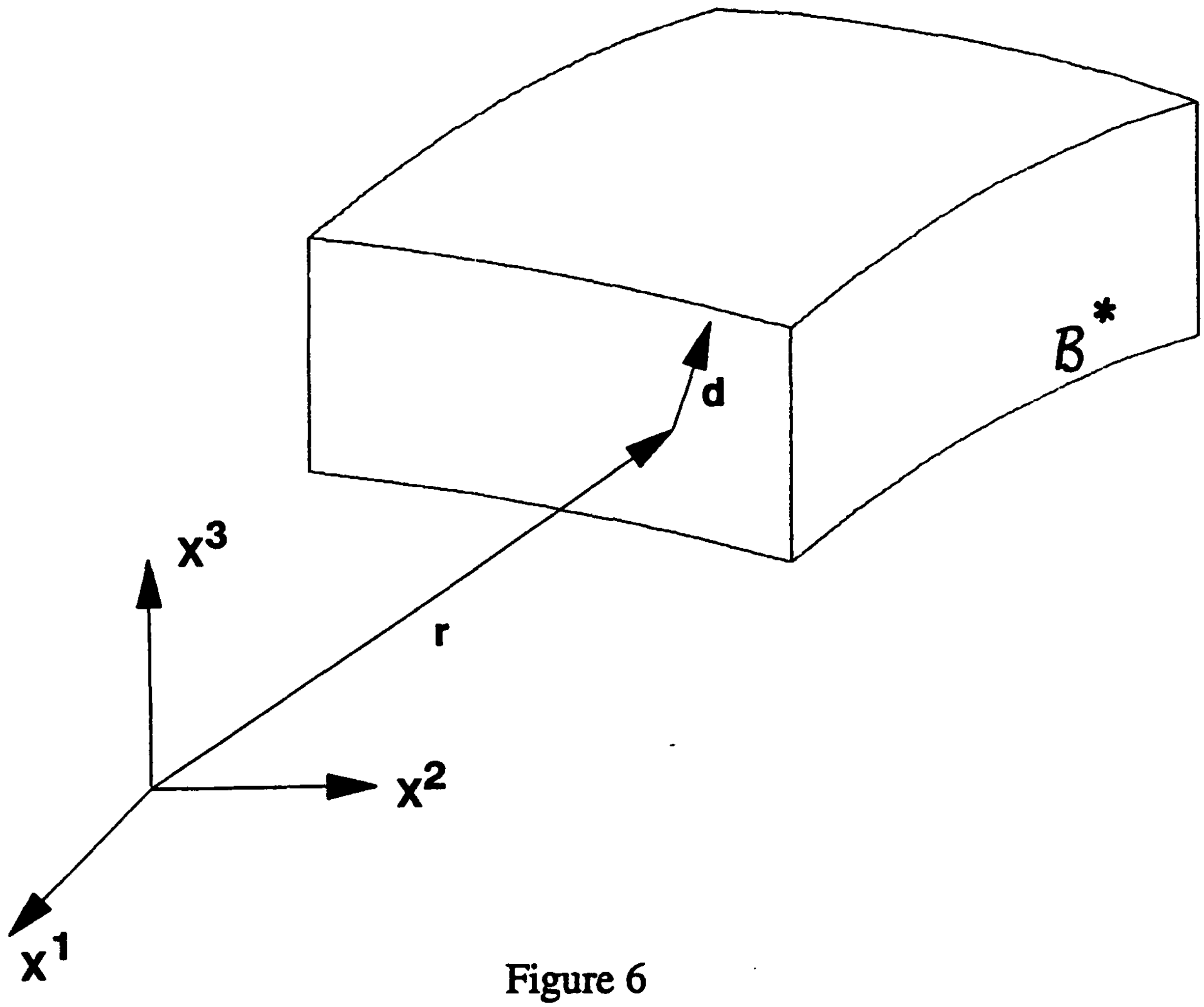

Figure 6
Macro-structure

It is to be emphasized that in the present discussion each Cosserat surface, i.e., micro-structure, is itself a three-dimensional shell-like body $\mathcal{B}^*$ consisting of two layers of different homogeneous materials. We also notice that the material points within each representative element $\mathcal{B}^*$ are regular particles in the sense of classical continuum mechanics while the material points of the macro-structure are endowed not only with an assigned mass density but also with a director. We will refer to the body $\mathcal{B}$ as composite laminate, macro-continuum or macro-structure

and to the body $\mathcal{B}^*$ as representative element, micro-continuum or micro-structure. Also, we will refer to particles of $\mathcal{B}$ as macro-particles or composite particles while the particles of the micro-structures will be referred to as micro-particles or simply particles (material points).

5. Definition of micro-structure

Within the context of three-dimensional classical continuum mechanics, consider a body $\mathcal{B}^*$ in the present configuration and let its boundary be a closed surface, denoted by $\partial\mathcal{B}^*$, and be composed of the following material surfaces:

a) The material surfaces

$$\begin{aligned} s_o &: \ \xi = 0 \\ & \qquad\qquad\qquad\qquad \xi_2 > 0 \\ s_2 &: \ \xi = \xi_2(\eta^\alpha) \end{aligned} \tag{5.1}$$

b) The material surface

$$s_\ell : \mathrm{f}(\eta^\alpha) = 0 \tag{5.2}$$

such that ξ = const. are closed smooth curves on the surface (5.2). We also consider a material surface of the form

$$s_1 : \ \xi = \xi_1(\eta^\alpha) \quad 0 < \xi_1 < \xi_2 \tag{5.3}$$

lying entirely between s_o and s_2.

Let the part of the body $\mathcal{B}^*$ which is enclosed by the surface s_o, s_1 and s_ℓ be designated by $\mathcal{B}_1^*$ and the part enclosed by the surfaces s_1, s_2 and s_ℓ be denoted by $\mathcal{B}_2^*$. The field quantities associated by the parts $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ will be designated by subscripts 1 and 2 when necessary.

6. Kinematics of micro- and macro-structures

We begin our development of the kinematical results by assuming that the position vector of a particle P^* of a representative element (micro-structure), i.e., $p^*(\eta^\alpha,\xi,\theta^3,t)$ in the present configuration has the form

$$\mathbf{p}^* = \mathbf{r}(\eta^\alpha,\theta^3,t) + \xi(\theta^3)\mathbf{d}(\eta^\alpha,\theta^3,t) \tag{6.1}$$

The dual of (6.1) in a reference configuration is given by

$$\mathbf{P}^* = \mathbf{R}(\eta^\alpha,\theta^3) + \xi(\theta^3)\mathbf{D}(\eta^\alpha,\theta^3) \tag{6.2}$$

The velocity vector $\mathbf{v}^*$ of the micro-structure at time t is given by

$$\mathbf{v}^* = \frac{\partial \mathbf{p}^*(\eta^\alpha,\xi,\theta^3,t)}{\partial t} = \dot{\mathbf{p}}^*(\eta^\alpha,\xi,\theta^3,t) \tag{6.3}$$

where a superposed dot denotes the material time derivative, holding η^i and θ^i fixed. From (6.1) and (6.3) we obtain

$$\mathbf{v}^* = \mathbf{v} + \xi\mathbf{w} \tag{6.4}$$

where

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{w} = \dot{\mathbf{d}} \tag{6.5}$$

are the velocity and the director velocity of the macro-structure, respectively. From (6.1) we have

$$\mathbf{g}_\alpha^* = \mathbf{a}_\alpha + \xi \frac{\partial \mathbf{d}}{\partial \eta^\alpha} \quad , \quad \mathbf{g}_3^* = \mathbf{d} \tag{6.6}$$

where $\mathbf{a}_\alpha$ are the surface base vectors of the surface s_o.

We recall that the velocity $\mathbf{v}$ and the director $\mathbf{d}$ are three-dimensional vectors and can be written as

$$\mathbf{v} = v_i \mathbf{g}^i = v^i \mathbf{g}_i \quad , \quad v_i = \mathbf{g}_i \cdot \mathbf{v} \quad , \quad v^i = g^{ij} v_j \tag{6.7}$$

$$\mathbf{d} = d_i \mathbf{g}^i = d^i \mathbf{g}_i \quad , \quad d_i = \mathbf{g}_i \cdot \mathbf{d} \quad , \quad d^i = g^{ij} d_j \tag{6.8}$$

The gradient of the director $\mathbf{d}$ may be obtained as follows:

$$\mathbf{d}_{i,} = \lambda_{ji} g^j = \lambda^j{}_i \mathbf{g}_j \tag{6.9}$$

For convenience we have introduced the notations

$$\lambda_{ij} = \mathbf{g}_i \cdot \mathbf{d}_{,j} = d_{i\,|\,j}$$

$$\lambda^i{}_j = \mathbf{g}^i \cdot \mathbf{d}_{,j} = d^i{}_{|\,j} \tag{6.10}$$

$$\lambda^i{}_j = g^{ik} \lambda_{kj}$$

Following the same procedure used we obtain

$$\mathbf{v}_{,i} = v_j \mathbf{g}^j = v^j{}_i \mathbf{g}_j \tag{6.11}$$

where

$$v_{ij} = \mathbf{g}_i \cdot \mathbf{v}_{,j} = v_{i\,|\,j}$$

$$v^i{}_j = \mathbf{g}^i \cdot \mathbf{v}_{,j} = v^i{}_{|\,j} \tag{6.12}$$

$$v^i_j = g^{ik} v_{kj}$$

Since v_{ij} is a second order covariant tensor, we may decompose it into its symmetric and its skew-symmetric parts, i.e.,

$$v_{ij} = v_{(ij)} + v_{[ij]} = \eta_{ij} + \omega_{ij} \tag{6.13}$$

where η_{ij} and ω_{ij} represent the symmetric and the skew-symmetric parts of v_{ij}, respectively. We may express $\dot{\mathbf{g}}_i$ in the form

$$\dot{\mathbf{g}}_i = \mathbf{v}_{,i} = (\eta_{ki} + \omega_{ki})\mathbf{g}^k \tag{6.14}$$

and

$$\mathbf{w} = \dot{\mathbf{d}} = \dot{d}_k \mathbf{g}^k + d^i(\omega_{ki} - \eta_{ki})\mathbf{g}^k \tag{6.15}$$

The gradient of the director velocity is given by

$$\mathbf{w}_{,i} = \dot{\mathbf{d}}_{,i} = \dot{\lambda}_{ki}\mathbf{g}^k + \lambda^j{}_i(\omega_{ki} - \eta_{kj})\mathbf{g}^k \tag{6.16}$$

We also introduce relative kinematical measures γ_{ij}, $\mathcal{K}_{ij}$ and γ_i such that

$$\gamma_{ij} = \frac{1}{2}(g_{ij} - G_{ij}) = \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) = \gamma_{ji} \tag{6.17}$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij} \tag{6.18}$$

and

$$\gamma_i = d_i - D_i \tag{16.19}$$

7. Basic field equations for the micro-structure

Making use of the theory of Cosserat (directed) surfaces after appropriate integration of the classical three-dimensional equations of motion, across the thickness of the micro-structure, we obtain the basic field equations for the shell-like micro-structure as follows:

$$\begin{aligned}
&\mathrm{a}: \quad \overline{(\hat{\rho} a^{1/2})}^{\bullet} = 0 \\
&\mathrm{b}: \quad \hat{\rho} a^{1/2}(\dot{\mathbf{v}} + y^1 \dot{\mathbf{w}}) = (\mathbf{N}^\alpha a^{1/2})_{,\alpha} + \hat{\rho}\hat{\mathbf{l}} a^{1/2} \\
&\mathrm{c}: \quad \hat{\rho} a^{1/2}(y^1 \dot{\mathbf{v}} + y^2 \dot{\mathbf{w}}) = (\mathbf{M}^\alpha a^{1/2})_{,\alpha} - \mathbf{m} a^{1/2} + \hat{\rho}\hat{\mathbf{l}} a^{1/2} \qquad (7.1) \\
&\mathrm{d}: \quad \mathbf{a}_\alpha \times \mathbf{N}^\alpha + \mathbf{d} \times \mathbf{m} + \mathbf{d}_{,\alpha} \times \mathbf{M}^\alpha = 0 \\
&\mathrm{e}: \quad \hat{\rho}(\dot{\hat{\varepsilon}}) = \mathbf{N}^\alpha \cdot \dot{\mathbf{v}}_{,\alpha} + \mathbf{M}^\alpha \cdot \mathbf{w}_{,\alpha} + \mathbf{m} \cdot \mathbf{w}
\end{aligned}$$

where the various field quantities appearing in (7.1) are

$\hat{\rho} = \hat{\rho}(\theta^\alpha,t)$:	The mass density of the micro-structure in the present configuration
$\nu = \nu(\theta^\alpha,t)$:	The outward unit normal to the boundary $\partial\hat{\mathcal{P}}$ of the micro-structure
$\mathbf{N}^\alpha = \mathbf{N}^\alpha(\theta^\alpha,t;\nu)$:	The resultant force per unit length of a curve in the present configuration
$\mathbf{M}^\alpha = \mathbf{M}^\alpha(\theta^\alpha,t;\nu)$:	The resultant couple per unit length of a curve in the present configuration
$\hat{\mathbf{f}} = \hat{\mathbf{f}}(\theta^\alpha,t)$:	The assigned force per unit mass of the micro-structure
$\hat{\mathbf{l}} = \hat{\mathbf{l}}(\theta^\alpha,t)$:	The assigned director force per unit mass of the micro-structure
$\mathbf{m} = \mathbf{m}(\theta^\alpha,t)$:	The intrinsic director force per unit area of the micro-structure
$y^\alpha = y^\alpha(\theta^\alpha)$:	The inertia coefficients
$\hat{\varepsilon} = \hat{\varepsilon}(\theta^\alpha,t)$:	The specific internal energy per unit mass of the micro-structure

8. Summary of basic principles for composite laminates

In this section we record the conservation laws for composite laminates and also collect various concepts/definitions that were introduced for composite laminates. With reference to the present configuration, these conservation laws are summarized below:

$$a: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho \, dv = 0$$

$$b: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{v} + y^1\mathbf{w})dv = \int_{\mathcal{P}} \rho\mathbf{b}dv + \int_{\partial\mathcal{P}} \mathbf{t}da$$

$$c: \quad \frac{d}{dt}\int_{\mathcal{P}} \rho(y^1\mathbf{v} + y^2\mathbf{w})dv = \int_{\mathcal{P}}(\rho\mathbf{c} - \mathbf{k})dv + \int_{\partial\mathcal{P}} \mathbf{s}da \tag{8.1}$$

$$d: \quad \frac{d}{dt}\int_{\mathcal{P}} \{\mathbf{r} \times (\mathbf{v} + y^1\mathbf{w}) + \mathbf{d} \times (y^1\mathbf{v} + y^2\mathbf{w})\}dv =$$

$$\int_{\partial\mathcal{P}} \rho(\mathbf{r} \times \mathbf{b} + \mathbf{d} \times \mathbf{c})dv + \int_{\partial\mathcal{P}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s})da$$

$$c: \quad \frac{d}{dt}\int \mathcal{P} \rho(\varepsilon + \mathcal{K})dv = \int_{\mathcal{P}} \rho(\mathbf{b} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w})dv + \int_{\partial\mathcal{P}} (\mathbf{t} \cdot \mathbf{v} + \mathbf{s} \cdot \mathbf{w})da$$

The first of (8.1) is the mathematical statement of conservation of mass, the second that of linear momentum principle, the third that of director momentum, the fourth is the principle of moment of momentum, and the fifth represents the balance of energy for composite laminates.

In (8.1), **r**,**d** denote the *position vector* and the *director* associated with a *composite particle*, respectively, while the velocity and the director velocity of the composite particle are given by **v** and **w**. The definition of the various field quantities in (8.1) and their relation to their

counterparts in micro-structure and the similar three dimensional quantities are reiterated below.

1) $\rho = \rho(\theta^i,t)$ is the *composite assigned mass density* in the present configuration given by

$$\rho g^{1/2} = \hat{\rho} a^{1/2} = \int_0^{\xi_2} \rho^* g^{*1/2} d\xi \tag{8.2}$$

where in (8.2) $\hat{\rho}$ is the mass density of the micro-structure, ρ^* is the classical 3-dimensional mass density, g is the determinant of the metric tensor g_{ij} associated with the composite coordinate system θ^i, g^* is the determinant of the metric tensor, g^*_{ij} associated with the micro-structure coordinate system $\eta^i = \{\eta^\alpha,\xi\} = \{\theta^\alpha,\xi\}$, a is the determinant of the two-dimensional (surface) metric tensor $a_{\alpha\beta}$ associated with the Cosserat surface (micro-structure).

We notice that the dimensions of ρ^* and $\hat{\rho}$ are mass per unit volume and mass per unit area, respectively. However, the dimension of ρ is the dimension of integrated mass per unit volume of the composite.

2) $\mathbf{b} = \mathbf{b}(\theta^i,t)$ is the *composite assigned body force density* per unit of ρ, given by

$$\rho g^{1/2} \mathbf{b} = \int_0^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* d\xi \tag{8.3}$$

where $\mathbf{b}^*$ is the classical 3-dimensional body force density. The dimension of $\mathbf{b}$ should be clear from (8.3).

3) $\mathbf{c} = \mathbf{c}(\theta^i,t)$ is the *composite assigned body couple density* per unit of ρ, given by

$$\rho g^{1/2} \mathbf{c} = \int_0^{\xi_2} \rho^* g^{*1/2} \mathbf{b}^* \xi d\xi \tag{8.4}$$

The dimension of **c** is clear from (8.4).

4) $\mathbf{t} = \mathbf{t}(\theta^i,t;\mathbf{n})$ is the *composite assigned stress vector* (per unit area of the composite) such that

$$\mathbf{t} = g^{-1/2}\mathbf{T}^i n_i \tag{8.5}$$

$$\mathbf{T}^A{}_{,A} = \mathbf{T}^i{}_{,i} = \int_0^{\xi_2} \mathbf{T}^{*i}{}_{,i} d\xi \tag{8.6}$$

$$\mathbf{T}^\alpha = \int_0^{\xi_2} \mathbf{T}^{*\alpha} d\xi = a^{1/2}\mathbf{N}^\alpha \tag{8.7}$$

$$\mathbf{T}^3{}_{,3} = \mathbf{T}^{*3}{}_{|\xi=\xi_2} - \mathbf{T}^{*3}{}_{|\xi=0} = \Delta\mathbf{T}^{*3} \tag{8.8}$$

where $\mathbf{T}^{*3}$ is the classical stress vector and $\mathbf{N}^\alpha$ is the resultant force of the micro-structure (i.e., Cosserat surface). We also recall that a comma on the left-hand side of (8.6) and (8.8) denotes partial differentiation with respect to θ^i. However, a comma on the right-hand side of (8.6) denotes partial differentiation with respect to $\eta^i = \{\eta^\alpha,\xi\}$.

5) $\mathbf{s} = \mathbf{s}(\theta^i,t;\mathbf{n})$ is the *composite assigned couple stress vector* per unit area of the composite such that

$$\mathbf{s} = g^{-1/2}\mathbf{S}^i n_i \tag{8.9}$$

$$\mathbf{S}^i{}_{,i} = \int_0^{\xi_2} \mathbf{T}^i{}_{,i}\xi d\xi \tag{8.10}$$

$$\mathbf{S}^\alpha = \int_0^{\xi_2} \mathbf{T}^{*\alpha}\xi d\xi = a^{1/2}\mathbf{M}^\alpha \tag{8.11}$$

$$S^3{}_{,3} = (\mathbf{T}^{*3}\xi)_{|\xi=\xi_2} - (\mathbf{T}^{*}\xi)_{|\xi=0} = \Delta(\mathbf{T}^{*3}\xi) \tag{8.12}$$

where $\mathbf{M}^\alpha$ is the resultant couple of the micro-structure (i.e., Cosserat surface) and the same remark as in (4) above holds for commas and partial differentiation.

6) $\mathbf{k} = \mathbf{k}(\theta^i,t)$ is the *composite assigned intrinsic (director) force*, per unit volume of the composite, given by,

$$g^{1/2}\mathbf{k} = a^{1/2}\mathbf{m} = \int_0^{\xi_2} \mathbf{T}^{*3} d\xi \tag{8.13}$$

where $\mathbf{m}$ is the intrinsic director force of the micro-structure (i.e., Cosserat surface).

7) $y^\alpha = y^\alpha(\theta^i)$ are the inertia coefficients which are independent of time and are given by

$$y^\alpha = \int_0^{\xi_2} \rho^* g^{*1/2} \xi^\alpha d\xi \tag{8.14}$$

8) $\varepsilon = \varepsilon(\theta^i,t)$ is the *composite assigned specific internal* energy per unit of ρ given by

$$\rho g^{1/2}\varepsilon = \hat{\rho} a^{1/2}\hat{\varepsilon} = \int_0^{\xi_2} \rho^* g^{*1/2} \varepsilon^* d\xi \tag{8.15}$$

where ε^* is the classical 3-dimensional specific internal energy and $\hat{\varepsilon}$ is the specific internal energy per unit $\hat{\rho}$ for the micro-structure (i.e., Cosserat surface).

9) $\mathcal{K} = \mathcal{K}(\theta^i,t)$ is the *composite assigned kinetic energy density* per unit of ρ and is given by

$$\mathcal{K} = \hat{\mathcal{K}} = \frac{1}{2}(\mathbf{v} \cdot \mathbf{v} + 2y^1 \mathbf{v} \cdot \mathbf{w} + y^2 \mathbf{w} \cdot \mathbf{w}) \tag{8.16}$$

where $\hat{\mathcal{K}}$ represents the kinetic energy density per unit $\hat{\rho}$ of the micro-structure (i.e., Cosserat surface). The momentum corresponding to the velocity $\mathbf{v}$ and the director momentum corresponding to $\mathbf{w}$ are given by

$$\rho \frac{\partial \mathcal{K}}{\partial \mathbf{v}} = \rho(\mathbf{v} + y^1 \mathbf{w}) \tag{8.17}$$

$$\rho \frac{\partial \mathcal{K}}{\partial \mathbf{w}} = \rho(y^1 \mathbf{v} + y^2 \mathbf{w}) \tag{8.18}$$

For simplicity in the rest of this development, when there is no possibility of confusion, we adopt the following simplified terminology:

ρ : "composite mass density"

b : "composite body force density"

c : "composite body couple density"

t : "composite stress vector"

s : "composite couple stress vector"

k : "composite intrinsic force"

ε : "composite specific internal energy"

$\mathcal{K}$: "composite kinetic energy"

We observe that the basic structures of $(8.1)_{a,b,c}$ and their forms are analogous to the corresponding conservation laws of the classical three-

dimensional continuum mechanics. Equation $(8.1)_c$ does not exist in the classical continuum mechanics whereas equations $(8.1)_{d,e}$, although they exist, they have simpler forms. It should be noted that the conservation laws (8.1) are consistent with the invariance requirements under superposed rigid body motions, which have wide acceptance in continuum mechanics. We also observe that concepts/quantities such as "body couple density, **c**," "couple stress vector **s**" and "intrinsic force **k**" are not admitted/defined in classical continuum mechanics.

9. Remarks on composite stress vector and composite couple stress vector

An important characteristic of the present theory is the introduction of the composite contact force,**t**, and the composite contact couple, **s**. It can be shown that **t** and **s** have the properties

$$\mathbf{t}(\theta^i,t\,;\,\mathbf{n}) = -\mathbf{t}(\theta^i,t\,;\,-\mathbf{n}) \qquad (9.1)$$

and

$$\mathbf{s}(\theta^i,t\,;\,\mathbf{n}) = -\mathbf{s}(\theta^i,t\,;\,-\mathbf{n}) \qquad (9.2)$$

where **n** is the outward unit normal to a surface within the composite. According to the results (9.1) and (9.2), the composite stress vector and the composite couple stress vector acting on opposite sides of the same surface at a given point within the composite laminates are equal in magnitude and opposite in direction. In addition, it can be demonstrated that $\mathbf{T}^i$ and $\mathbf{S}^i$ are expressible as

$$\mathbf{T}^i = g^{1/2}\tau^{ij}\mathbf{g}_j \qquad (9.3)$$

and

$$\mathbf{S}^i = g^{1/2}s^{ij}\mathbf{g}_j \qquad (9.4)$$

where τ^{ij} and s^{ij} are contravariant components of the *composite stress tensor* and the *composite couple stress tensor.* It can also be shown that $\mathbf{T}^3$ represent the effect of interlaminar stresses τ^{3j} (j = 1,2,3) through (9.3).

10. Basic field equations for composite laminates

The basic field equations for composite laminates follows from $(8.1)_a$ to $(8.1)_c$ and are given by

$$
\begin{aligned}
&a : \quad \dot{\rho} + \frac{\dot{g}}{2g}\rho = 0 \\
&b : \quad \mathbf{T}^i{}_{,i} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \\
&c : \quad \mathbf{S}^i{}_{,i} + g^{1/2}(\rho\mathbf{c} - \mathbf{k}) = \rho g^{1/2}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \qquad (10.1) \\
&d : \quad \mathbf{g}_i \times \mathbf{T}^i + \mathbf{d}_{,i} \times \mathbf{S}^i + g^{1/2}\mathbf{d} \times \mathbf{k} = 0 \\
&e : \quad \rho g^{1/2}\dot{\varepsilon} = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} = g^{1/2}P
\end{aligned}
$$

where P is the mechanical power density (per element of volume) of the composite and is given by

$$g^{1/2}P = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w} \qquad (10.2)$$

The basic field equations (10.1) are both simple and elegant in form. In practice, we usually work with the components of the various fields. Hence, we now proceed to deduce the basic field equations in tensor components. We introduce the contravariant and covariant components of acceleration (α^i,α_i), director acceleration (β^i,β_i), body force (b^i,b_i), body couple (c^i,c_i), and those of intrinsic force (k^i,k_i) as follows.

$$
\begin{aligned}
&\dot{\mathbf{v}} = \alpha^i\mathbf{g}^i = \alpha_i\mathbf{g}^i \;,\; \dot{\mathbf{w}} = \beta^i\mathbf{g}_i = \beta_i\mathbf{g}^i \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (10.3) \\
&\mathbf{b} = b^i\mathbf{g}_i = b_i\mathbf{g}^i \;,\; \mathbf{c} = c^i\mathbf{g}_i = d_i\mathbf{g}^i \;,\; \mathbf{k} = k^i\mathbf{g}_i = k_i\mathbf{g}^i
\end{aligned}
$$

The basic field equations (10.1) when expressed in component forms will reduce to

$$a: \quad \dot{\rho} + \frac{\dot{g}}{2g}\rho = 0$$

$$b: \quad \tau^{ij}{}_{|i} + \rho b^j = \rho(\alpha^j + y^1\beta^j)$$

$$c: \quad s^{ij}{}_{|i} + (\rho c^j - k^j) = \rho(y^1\alpha^j + y^2\beta^j) \tag{10.4}$$

$$d: \quad \varepsilon_{ijn}(\tau^{ih} + d^i{}_{|m}s^{mj} + d^ik^j) = \varepsilon_{ijn}(\tau^{ij} - s^{mi}\dot{\lambda}_m - k^id^j) = 0$$

$$e: \quad \rho\dot{\varepsilon} = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^iw_i = P$$

while the expression for mechanical power density P takes the form

$$\rho\dot{\varepsilon} = \tau^{ij}v_{j|i} + s^{ij}w_{j|i} + k^iw_i = P \tag{10.5}$$

With reference to $(10.4)_d$ we observe that the symmetry of the stress tensor is not valid. However, because ε_{ijk} is skew-symmetric with respect to i and j, it follows that the quantity in the parentheses in $(10.4)_d$ must be symmetric with respect to i and j. Hence the quantities

$$\tau'^{ij} = \tau^{ij} - s^{mj}\lambda^j m - k^id^j \tag{10.6}$$

are symmetric. We call τ'^{ij} the *composite assigned symmetric stress tensor* or simply the *composite symmetric stress tensor.* We notice that in the absence of the director, i.e.,

$$\mathbf{d} = 0 \quad \text{or} \quad d^i = 0$$

the composite symmetric stress tensor τ^{ij} reduces to the classical stress tensor. It can be shown that in the absence of the micro-structure and the director the basic field equations (10.4) as well as the expressions for power reduce to those of classical continuum mechanics.

11. Constitutive equations for nonlinear elastic composite laminates

Within the scope of the theory developed in previous sections, we discuss the constitutive relations for elastic composite laminates in the presence of finite deformation and in the context of purely mechanical theory.

We recall that a material is defined by a constitutive assumption which characterizes the mechanical behavior of the medium. The constitutive assumption places a restriction on the processes which are admissible in a body — here the composite laminate.

We recall that in the three-dimensional theory of classical (non-polar) continuum mechanics and within the context of purely mechanical theory the constitutive relation for the specific internal energy and the stress tensor of an elastic body can be expressed as follows

$$\psi^* = \psi^*(\gamma_{ij}) \tag{11.1}$$

$$\tau^{*ij} = \rho^* \frac{\partial \psi^*}{\partial \gamma^*_{ij}} \tag{11.2}$$

We now proceed to deduce the counterparts of the above results for an elastic composite laminate. To this end, we first recall the expression for γ^*_{ij}, i.e.,

$$\gamma^*_{ij} = \gamma^*_{ji} = \frac{1}{2} (\mathbf{g}^*_i \cdot \mathbf{g}^*_j - \mathbf{G}^*_i \cdot \mathbf{G}^*_j)$$

and then observe the following relations*:

$$\frac{\partial \psi^*}{\partial \mathbf{g}_k^*} = \frac{\partial \psi^*}{\partial \gamma_{ki}^*} \mathbf{g}_i^*$$

$$\frac{\partial \psi^*}{\partial \mathbf{a}_\alpha} = \frac{\partial \psi^*}{\partial \gamma_{3i}^*} \mathbf{g}_i^* \tag{11.3}$$

$$\frac{\partial \psi^*}{\partial \mathbf{d}_{,\alpha}} = \frac{\partial \psi^*}{\partial \gamma_{\alpha i}^*} \mathbf{g}_i^* \xi$$

Considering the constitutive equations for the stress components $\tau^{*\alpha i}$, i.e.,

$$\tau^{*\alpha j} = \rho^* \frac{\partial \psi^*}{\partial \gamma_{\alpha j}^*}$$

and recalling the formula

$$\mathbf{T}^{*\alpha} = g^{*1/2} \tau^{*\alpha j *} \mathbf{g}_j^*$$

and

$$\mathbf{T}^\alpha = \int_o^{\xi_2} \mathbf{T}^{*\alpha} d\xi$$

---

* Operators of the form $\dfrac{\partial f}{\partial \mathbf{x}}$ where f is a scalar valued function of a vector $\mathbf{x} = x^i \mathbf{g}_i = x_i \mathbf{g}^i$ were defined earlier. The component form of this operator which is in fact the gradient operator (derivative operator) is given by

$$\frac{df}{d\mathbf{x}} = \frac{\partial f}{\partial x_i} \mathbf{g}_i = \frac{\partial f}{\partial x^i} \mathbf{g}^i$$

we obtain

$$\mathbf{T}^{\alpha} = \int_{o}^{\xi_2} \mathbf{T}^{*\alpha} d\xi = \frac{\partial}{\partial \mathbf{g}_{\alpha}} \int_{o}^{\xi_2} \rho^* g^{*1/2} \psi^* d\xi \tag{11.4}$$

Next, recall the formula

$$\mathbf{S}^{\alpha} = \int_{o}^{\xi_2} \mathbf{T}^{*\alpha} \xi d\xi$$

and write

$$\mathbf{S}^{\alpha} = \int_{o}^{\xi_2} \mathbf{T}^{*\alpha} \xi d\xi = \frac{\partial}{\partial \mathbf{d}_{,\alpha}} \int_{o}^{\xi_2} \rho^* g^{*1/2} \psi^* d\xi \tag{11.5}$$

Now recall the expression

$$g^{1/2}\mathbf{k} = a^{1/2}\hat{\mathbf{k}} = \int_{o}^{\xi_2} \mathbf{T}^{*3}\, d\xi$$

and write

$$g^{1/2}\mathbf{k} = \int_{o}^{\xi_2} g^{*1/2} \tau^{*3j} \mathbf{g}_j^* d\xi = \frac{\partial}{\partial \mathbf{d}} \int_{o}^{\xi_2} \rho^* g^{*1/2} \psi^* d\xi \tag{11.6}$$

We notice that interlaminar stress vector $\mathbf{T}^3$ acts as an applied contact force for the micro-structure. Hence, in general, the constitutive relation for $\mathbf{T}^3$ should be specified directly. This means, in general, $\mathbf{T}^3$ unlike $\mathbf{T}^{\alpha}$ may not be obtained from the strain energy of the constituents.

Consider now the expression

$$\overline{\psi} = \frac{1}{\rho g^{1/2}} \int_{o}^{\xi_2} \rho^* g^{*1/2} \psi^*(\mathbf{g}_{\alpha}^*, \mathbf{d}, \mathbf{d}_{,\alpha}) d\xi + \frac{1}{\rho g^{1/2}} \hat{\psi}(\mathbf{g}_3) \tag{11.7}$$

where the arguments of ψ^* have been defined before and where $\hat{\psi}(g_3)$ is a suitable function of $\mathbf{g}_3$. Clearly, in view of kinematical relations of section 4, the function ψ can be regarded as a function of the variables $\mathbf{g}_i$,$\mathbf{d}$ and $\mathbf{d}_{,i}$. Therefore, the constitutive equations for composite laminate will be given by

$$\psi = \bar{\psi}(\mathbf{g}_i,\mathbf{d},\mathbf{d}_{,\alpha}) \tag{11.8}$$

$$\mathbf{T}^i = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{g}_i} \tag{11.9}$$

$$\mathbf{S}^i = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{d}_{,i}} \tag{11.10}$$

$$g^{1/2}\mathbf{k} = \rho g^{1/2} \frac{\partial \bar{\psi}}{\partial \mathbf{d}} \tag{11.11}$$

where $\mathbf{S}^3$ vanishes identically since $\bar{\psi}$ is not a function of $\mathbf{d}_{,3}$. The above constitutive equations are subject to condition $(10.1)_d$.

For completeness, we also record the component forms of (11.9). It is clear that the function $\bar{\psi}$ may be rewritten as

$$\psi = \bar{\psi}(\mathbf{g}_i,\mathbf{d},\mathbf{d}_{,\alpha}) = \bar{\psi}(\mathbf{r}_{,i},\mathbf{d},\mathbf{d}_{,\alpha}) = \tilde{\psi}(r_{m|i},d_m,d_{m|\alpha}) \tag{11.12}$$

With the help of the expressions for $\mathbf{T}^i$ and $\mathbf{S}^i$ and the gradient of a scalar valued function of a vector, we obtain

$$\tau^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial \tau_{j|i}} \tag{11.13}$$

$$s^{ij} = \rho \frac{\partial \tilde{\psi}}{\partial d_{j|i}} \tag{11.14}$$

$$k^i = \rho \frac{\partial \tilde{\psi}}{\partial d_i} \tag{11.15}$$

These are the component forms of the constitutive equations for τ^{ij}, s^{ij} and k^i along with the condition $(10.4)_d$, i.e.,

$$\varepsilon_{ijn}\{\tau^{ij} - s^{mi}\lambda^{\,j}_{m} - k^i d^j\} = 0$$

which is imposed by the principal of the moment of momentum of the composite laminate and must be satisfied by the response function $\tilde{\psi}$.

Before proceeding further, we obtain an alternative form of constitutive equations. To this end we consider the expression for mechanical power, i.e.,

$$g^{1/2}P = \mathbf{T}^i \cdot \mathbf{v}_{,i} + \mathbf{S}^i \cdot \mathbf{w}_{,i} + g^{1/2}\mathbf{k} \cdot \mathbf{w}$$

and by taking advantage of various kinematical relations we obtain

$$P = (\tau^{ij} - s^{mj}\lambda^{\,i}_{m} - k^j d^i)\eta_{ji} + s^{ij}\dot{\lambda}_{ji} + k^i \dot{d}_i + (\tau^{ij} + s^{mj}\lambda^{\,i}_{m} + k^j d^i)\omega_{ji} \tag{11.16}$$

The last term on the right hand side of (11.16) is a produce to a symmetric and a skew-symmetric tensor component; hence it vanishes identically and we obtain

$$P = (\tau^{ij} - s^{mj}\lambda^{\,i}_{m} - k^j d^i)\eta_{ji} + s^{ij}\dot{\lambda}_{ji} + k^i \dot{d}_i \tag{11.17}$$

We now recall the expression for the symmetric composite stress tensor τ'^{ij} and substitute for τ^{ij} in (11.17) to obtain

$$P = \tau'^{ij}\eta_{ij} + s^{ij}\dot{\lambda}_{ji} + k^i \dot{d}_i \tag{11.18}$$

Recall the kinematical variables

$$\gamma_{ij} = \frac{1}{2} (\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) = \frac{1}{2} (g_{ij} - G_{ij})$$

$$\mathcal{K}_{ij} = \lambda_{ij} - \Lambda_{ij}$$

$$\gamma_i = d_i - D_i$$

We may write

$$\dot{\gamma}_{ij} = \frac{1}{2} \dot{g}_{ij} = \frac{1}{2} (2\eta_{ij}) = \eta_{ij}$$

$$\dot{\mathcal{K}}_{ij} = \dot{\lambda}_{ij} \tag{11.19}$$

$$\dot{\gamma}_i = \dot{d}_i$$

The expression of power (11.17) in terms of the kinematical variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i is

$$\rho\dot{\varepsilon} = \tau'^{ij}\dot{\gamma}_{ij} + s^{ij}\dot{\mathcal{K}}_{ij} + k^i\dot{\gamma}_i = P \tag{11.20}$$

Rewriting the $\tilde{\psi}$ as a function of the variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i, i.e.,

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_{ij}) \tag{11.21}$$

we obtain

$$\dot{\varepsilon} = \frac{\partial\psi}{\partial\gamma_{ij}} \dot{\gamma}_{ij} + \frac{\partial\psi}{\partial\mathcal{K}_{ij}} \dot{\mathcal{K}}_{ij} + \frac{\partial\psi}{\partial\gamma_i} \dot{\gamma}_i \tag{11.22}$$

From (11.20) and (11.22) we obtain

$$(\tau'^{ij} - \rho \frac{\partial \psi}{\partial \gamma_{ij}})\dot{\gamma}_{ij} + (s^{ij} - \rho \frac{\partial \psi}{\partial \mathcal{K}_{ij}})\dot{\mathcal{K}}_{ij} + (k^{i} - \rho \frac{\partial \psi}{\partial \gamma_{i}})\dot{\gamma}_{i} = 0 \quad (11.23)$$

Then by the usual procedure in continuum mechanics we obtain

$$\tau'^{ij} = \rho \frac{\partial \psi}{\partial \gamma_{ij}} \quad (11.24)$$

$$s^{ij} = \rho \frac{\partial \psi}{\partial \mathcal{K}_{ij}} \quad (11.25)$$

$$k^{i} = \rho \frac{\partial \psi}{\partial \gamma_{i}} \quad (11.26)$$

12. A constrained theory of composite laminates

So far our development of the continuum theory has been general and without any restriction/condition placed on the kinematical variables. Therefore the field equations and the constitutive relations are applicable to any elastic composite laminate. We did not previously introduce any kinematical constraints to keep the theory general enough so that it could be utilized for various physical situations. We now turn to the development of a constrained theory of our continuum model. First we derive a set of constraint equations for the composite laminate. We then proceed to obtain the relevant response functions induced by the constraint. Finally we obtain a set of field equations in terms of the displacement and effected by the presence of the constraints.

We impose the condition that plies of the composite laminate do not separate from or slide over each other during the motion of the composite laminate. This means the displacement vector of the material points thoughtout the body, including at the interface, must be single valued. Hence we require

$$\frac{\mathbf{r}(\theta^\alpha,\theta^3+\Delta\theta^3)-\mathbf{r}(\theta^\alpha,\theta^3)}{\xi_2}=\mathbf{d}(\theta^\alpha,\theta^3) \tag{12.1}$$

In the limit when $\xi_2 \to 0$ and $\mathbf{r}(\theta^\alpha,\theta^3+\Delta\theta^3) \to \mathbf{r}(\theta^\alpha,\theta^3)$ we obtain

$$\mathbf{g}_3=\mathbf{r}_{,3}=\mathbf{d} \tag{12.2}$$

where we have made use of the assumption

$$\Delta\theta^3=\xi_2 \tag{12.3}$$

Expression (12.2) implies the following constraint condition

$$\dot{\mathbf{g}}^\alpha\cdot\mathbf{d}+\mathbf{g}^\alpha\cdot\mathbf{w}=0 \quad (\alpha=1,2)$$

$$d^i \mathbf{g}^\alpha \cdot \mathbf{v}_{,i} - \mathbf{g}^\alpha \cdot \mathbf{w} = 0 \quad (\alpha = 1,2)$$

$$d_m g^{im} g^{j\alpha} v_{j|i} - g^{i\alpha} w_i = 0 \tag{12.4}$$

For a composite laminate with constraints we assume that each of the functions $\mathbf{T}^i$,$\mathbf{S}^i$ and $\mathbf{k}^i$ are determined to within an additive constraint response so that

$$\mathbf{T}^i = \tilde{\mathbf{T}}^i + \hat{\mathbf{T}}^i$$

$$\mathbf{S}^i = \tilde{\mathbf{S}}^i + \hat{\mathbf{S}}^i \tag{12.5}$$

$$\mathbf{k} = \tilde{\mathbf{k}} + \hat{\mathbf{k}}$$

where

$$\hat{\mathbf{T}}^i \ , \ \hat{\mathbf{S}}^i \ , \ \hat{\mathbf{k}} \tag{12.6}$$

are specified by constitutive equations and

$$\tilde{\mathbf{T}}^i \ , \ \tilde{\mathbf{S}}^i \ , \ \tilde{\mathbf{k}} \tag{12.7}$$

which represent the response due to constraints (12.4), are arbitrary functions of θ^i,t, are workless and independent of the kinematical variables $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$. Thus, recalling the expression (10.2) for mechanical power, we set

$$\tilde{\mathbf{T}}^i \cdot \mathbf{v}_{,i} + \tilde{\mathbf{S}}^i \cdot \mathbf{w}_{,i} + g^{1/2} \hat{\mathbf{k}} \cdot \mathbf{w} = 0 \tag{12.8}$$

This must hold for all values of the variables $\mathbf{v}_{,j}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$ subject to the constraint condition (12.4). Multiplying (12.4) by the Lagrange

multipliers δ_α (α = 1,2) and subtracting the results from (12.8), we obtain

$$(\tilde{\mathbf{T}}^i - \delta_\alpha d^i \mathbf{g}^\alpha) \cdot \mathbf{v}_{,i} + \tilde{\mathbf{S}}^i \cdot \mathbf{w}_{,i} + (g^{1/2}\tilde{\mathbf{k}} + \delta_\alpha \mathbf{g}^\alpha) \cdot \mathbf{w} = 0 \tag{12.9}$$

From (12.9) and the fact that $\tilde{\mathbf{T}}^i$, $\tilde{\mathbf{S}}^i$ and $\tilde{\mathbf{k}}$ are independent of $\mathbf{v}_{,i}$, $\mathbf{w}_{,i}$ and $\mathbf{w}$ it follows that

$$\tilde{\mathbf{T}}^i = \delta_\alpha d^i \mathbf{g}^\alpha \tag{12.10}$$

$$\tilde{\mathbf{S}}^i = 0 \tag{12.11}$$

$$g^{1/2}\tilde{\mathbf{k}} = - \delta_\alpha \mathbf{g}^\alpha \tag{12.12}$$

Expressions (12.10) to (12.11) represent the constraint response induced by the constraint equations (12.4). Substituting these into linear momentum equation $(10.1)_b$ and the director momentum equation $(10.1)_c$, we obtain

$$[\hat{\mathbf{T}}^i + \delta_\alpha d^i \mathbf{g}^\alpha]_{,i} + \rho g^{1/2}\mathbf{b} = \rho g^{1/2}(\dot{\mathbf{v}} + y^1\dot{\mathbf{w}}) \tag{12.13}$$

and

$$\hat{\mathbf{S}}^i{}_{,i} + \rho g^{1/2}\mathbf{c} - [g^{1/2}\hat{\mathbf{k}} - \delta_\alpha \mathbf{g}^\alpha] = \rho g^{1/2}(y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}}) \tag{12.14}$$

Introducing the following temporary variables $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ by

$$\hat{\mathbf{b}} = \mathbf{b} - (\dot{\mathbf{v}} + y^1\dot{\mathbf{w}})$$

and (12.15)

$$\hat{\mathbf{c}} = \mathbf{c} - (y^1\dot{\mathbf{v}} + y^2\dot{\mathbf{w}})$$

from (12.13) and (12.14) we obtain

$$\rho g^{1/2}\hat{\mathbf{b}} + \hat{\mathbf{T}}^{i}{}_{,i} - (\rho g^{1/2}d^{i}\hat{\mathbf{c}} + d^{i}\hat{\mathbf{S}}^{i}{}_{,i} - g^{1/2}d^{i}\mathbf{k})_{,i} = 0 \tag{12.16}$$

$$\rho g^{1/2}\mathbf{d}\cdot\hat{\mathbf{c}} + \mathbf{d}\cdot\hat{\mathbf{S}}^{j}{}_{j} - g^{1/2}\mathbf{d}\cdot\hat{\mathbf{k}} = 0 \tag{12.17}$$

Recalling that $\hat{\mathbf{T}}^{i,}$ $\hat{\mathbf{S}}^{i}$ and $\hat{k}$ are specified as functions of various kinematical variables, it is clear that the system of equations (12.10) to (12.12), (12.16) and (12.17) represent two equation? for the determination of the primary unknowsn $\mathbf{v}$ (or $\mathbf{r}$) and $\mathbf{d}$.

The counterparts of (12.16) and (12.17) in component form are given by

$$\overline{\tau}^{ij} = \lambda_{\alpha} d^{i} g^{j\alpha} \tag{12.18}$$

$$\overline{s}^{ij} = 0 \tag{12.19}$$

$$\overline{k}^{i} = -\,\lambda_{\alpha} g^{i\alpha} \tag{12.20}$$

and

$$\rho\hat{b}^{j} + \hat{\tau}^{ij}{}_{|i} - (\rho d^{i}\hat{c}^{j} + \rho d^{i}\hat{s}^{mj}{}_{|m} - d^{i}\hat{k}^{j})_{|i} = 0 \tag{12.21}$$

$$\rho d_{j}\hat{c}^{j} + d_{j}\hat{s}^{ij}{}_{|i} - d_{j}\hat{k}^{j} = 0 \tag{12.22}$$

Again recalling that $\hat{\tau}^{ij}$, $\hat{s}^{ij}$ and $\hat{k}^{i}$ are specified, by constitutive equations, as functions of relevant kinematic variables.

13. Linearized field equations

Recalling the usual linearization procedure of the previous section and avoiding the introduction of additional notations, we now regard deformation measures as infinitesimal quantities of $O(\varepsilon)$. We omit the details since it is a straightforward calculation and merely record the linearized version of the equations of motion as follows:

$$\tau^{i}{}_{j\,|\,i} + \rho_o b_j = \rho_o(\ddot{u}_j + y^1\ddot{\delta}_j) \tag{13.1}$$

$$s^{i}{}_{j\,|\,i} + (\rho_o c_j - k_j) = \rho_o(y^1\ddot{u}_j + y^2\ddot{\delta}_j) \tag{13.2}$$

$$\varepsilon_{ijn}\{\tau^{ij} + s_{mj}\Lambda^{i}{}_{m} + D^i k^j\} = 0 \tag{13.3}$$

where the vertical bar in (13.1) and (13.3) and the rest of this section denotes covariant differentiation with respect to G_{ij}. We also note that all quantities are now referred to the base vectors G_i of the reference configuration.

Moreover, upon linearization we obtain

$$\tau'_{ij} = \tau^{ij} - \Lambda^{i}{}_{m}s^{mj} - D^i k^j \tag{13.4}$$

In the light of the assumptions stated above and expression (13.4), the energy equation takes the form

$$\rho_o\dot{\varepsilon} = \tau'^{ij}\dot{\gamma}_{ji} + s^{ij}\dot{\mathcal{K}}_{ji} + k^i\dot{\gamma}_i = P \tag{13.5}$$

14. Linear constitutive relations for elastic composite laminates

This section is concerned with the derivation of the constitutive relations for a composite laminate in terms of those of its constituents. In what follows we assume that each of the constituents of the laminated composite is a homogeneous isotropic elastic material. We recall that within the scope of the linear theory all kinematical variables are referred to the reference configuration. Previously we showed that the strain energy function, ψ may be written as

$$\psi = \psi(\gamma_{ij}, \mathcal{K}_{ij}, \gamma_i) \tag{14.1}$$

We assume that in the case of the linear theory ψ is given by a quadratic function of the infinitesimal kinematical variables γ_{ij}, $\mathcal{K}_{ij}$ and γ_i. We also recall that after systematic linearization of the expression for power, we obtained for the linear theory

$$\rho_o \dot{\varepsilon} = \tau^{ij} \dot{\gamma}_{ji} + s^{ij} \dot{\mathcal{K}}_{ji} + k^i \dot{\gamma}_i = P \tag{14.2}$$

and

$$\tau^{ij} = \rho_o \frac{\partial \psi}{\partial \gamma_{ij}}$$

$$s^{ij} = \rho_o \frac{\partial \psi}{\partial \mathcal{K}_{ji}} \tag{14.3}$$

$$k^i = \rho_o \frac{\partial \psi}{\partial \gamma_i}$$

The relationship between the strain energy function ψ, per unit mass of the composite, and those of the constituents is given by

$$\psi = \frac{1}{\rho_o G^{1/2}} \int_o^{\xi_2} \rho_o^* G^{*1/2} \psi^d \xi \tag{14.4}$$

or

$$\rho_o \psi = \int_o^{\xi_2} \nu(\rho_o^* \psi^*) d\xi = \int_o^{h_2} \mu(\rho_o^* \psi^*) d\xi$$

$$= \int_o^{h_1} \mu(\rho_{o1}^* \psi_1^*) d\xi + \int_{h_1}^{h_2} \mu(\rho_{o2}^* \psi_2^*) d\xi \tag{14.5}$$

where ρ_{o1}^* and ρ_{o2}^* denote mass densities of the constituents $\hat{\mathcal{B}}_1^*$ and $\hat{\mathcal{B}}_2^*$. We recall that in three-dimensional linear theory of elasticity we have

$$\rho_o^* \psi^* = \frac{1}{2} E_{mn}^{*ij} \gamma_{ij}^* \gamma^{*mn} \tag{14.6}$$

and

$$\tau^{*ij} = E_{mn}^{*ij} \gamma^{*mn} \tag{14.7}$$

We also recall that for isotropic elastic materials we have

$$E_{mn}^{*ij} = \lambda^* G^{*ij} G_{mn}^* + \mu^* (\delta^{im} \delta^j{}_n + \delta^i{}_n \delta^j{}_m) \tag{14.8}$$

$$\tau^{*ij} = \mu^* (G^{*im} G^{*jn} + G^{*in} G^{*jm} + \frac{2\nu^*}{1-2\nu^*} G^{*ij} G^{*mn}) \gamma_{mn}^* \tag{14.9}$$

$$\lambda^* = \frac{2\nu^*}{1-2\nu^*} \mu^* \tag{14.10}$$

For an explicit set of constitutive relations the integration on the right

hand side of (14.5) must be carried out using (14.6) for $\rho_0^* \psi^*$. Here we remark that as in the case of two dimensional theories of continuum mechanics (such as plates and shells), except possibly in very special cases, it appears to be extremely difficult to calculate the function ψ in (14.2) form the strain energy function ψ^* of the classical three dimensional theory. In the case of composite materials this becomes more complicated due to the existence of two (or more) materials.

Alternatively, in order to provide constitutive relations in which the coefficients are related to elastic constants of the constituents we can make use of the so-called specific Gibbs energy function. This method proves to be more convenient for the derivation of the linear constitutive equations for a composite laminate and will be described in the next section.

15. Linear constitutive relations for composite laminates: An alternative procedure

In this section we introduce an alterntive procedure for the derivation of the linear constitutive equations for a composite laminate. The explicit integration of (14.4) in most cases becomes exceedingly difficult. Here we provide an alternative appproach for explicit derivation of the constitutive relations (for the linear theory of a composite laminate) in which the coefficients are related to the elastic constants of the continuum.

We recall that the constitutive equations of the classical linear theory of elasticity in the context of purely mechanical theory may be expressed in terms of the three-dimensional specific Gibbs free energy function, say ϕ^*, in the form

$$\gamma^*_{ij} = -\rho^*_o \frac{\partial \phi^*}{\partial \tau^{*ij}} \tag{15.1}$$

where γ^*_{ij} is the infinitesimal strain and where ϕ^* and ψ^* are related through

$$\phi^* = \phi^*(\tau^{*ij}) = \psi^*(\gamma^*_{ij}) - \frac{1}{\rho^*_o} \tau^{*ij}\gamma^*_{ij} \tag{15.2}$$

and ϕ^* and ψ^* are quadratic functions of their arguments and both also depend on the reference values of G^*_{ij}. It may be noted that the function ϕ^* defined by (15.2) is the negative of the expression for the complimentary energy density. We now recall thst the Gibbs function ϕ^* for an initially homogeneous and isotropic material can be expressed as

$$\rho^*_o\phi^* = \{-\frac{1+\nu^*}{2E^*} G^*_{im}G^*_{jn} + \frac{\nu^*}{2E^*} G^*_{ij}G^*_{mn}\} \tau^{*ij}\tau^{*mn} \tag{15.3}$$

where G^*_{ij} is the initial metric tensor, E^* is Young's modulus of elasticity

and ν^* is Poisson's ratio.

Within the scope of the linear theory and corresponding to (12.6) we define a *composite Gibbs free energy* (or a "composite complementary energy") ϕ as follows:

$$\rho_o G^{1/2}\psi = \int_o^{\xi_2} \rho_o^* G^{*1/2}\phi^* d\xi \tag{15.4}$$

From (15.2), by integration with respect to ξ between zero and ξ^2 and making use of (14.6), (15.2) and (15.4), we obtain

$$\phi = \psi - \frac{1}{\rho_o}\int_o^{\xi_2} \nu\tau^{*ij}\gamma_{ij}^* d\xi \tag{15.5}$$

By making use of the expressions for τ^{*ij}, γ_{ij}^*, the expressions for various resultants and the kinematic assumptions for **R** and **D**, we can express the integral in (13.6) in terms of the various resultants and their corresponding relative kinematic measures. However, as before the constitutive relations for the interlaminar stress vectors $\mathbf{T}^i$ should be specified directly. Keeping this and expressions (15.2) and (15.5) in mind, we assume the existence of a Gibbs free energy function ϕ, such that

$$\rho_o\phi = \rho_o\bar{\phi}(\bar{\tau}^{ij}, s^{ij}, k^i) = \rho_o\psi - \{\bar{\tau}^{ij}\gamma_{ij} + s^{ij}\mathcal{K}_{ij} + k^i\gamma_i\} \tag{15.6}$$

From (15.6) and the expression for power we obtain

$$(\gamma_{ij} + \rho_o\frac{\partial\bar{\phi}}{\partial\bar{\tau}^{ij}})\dot{\bar{\tau}}^{ij} + (\mathcal{K}_{ij} + \rho_o\frac{\partial\bar{\phi}}{\partial s^{ij}})\dot{s}^{ij} + (\gamma_i + \rho_o\frac{\partial\bar{\phi}}{\partial k^i})\dot{k}^i = 0 \tag{15.7}$$

where we have assumed the rates $\dot{\bar{\tau}}^{ij}$, $\dot{s}^{ij}$ and $\dot{k}^i$ are all independent and their coefficients are rate independent. From (15.7) it follows

$$\gamma_{ij} = -\rho_o \frac{\partial \bar{\phi}}{\partial \tau^{ij}}$$

$$\mathcal{K}_{ij} = -\rho_o \frac{\partial \bar{\phi}}{\partial s^{ij}} \tag{15.8}$$

$$\gamma_i = -\rho_o \frac{\partial \bar{\phi}}{\partial k^i}$$

We note that the relationship between the Gibbs energy function ϕ, per unit mass of the composite, and those of the constituents is given by

$$\rho_o \phi = \int_o^{\xi_2} \nu(\rho_o^* \phi^*) d\xi = \int_o^{h_2} \mu(\rho_o^* \phi^*) d\xi$$

$$= \int_o^{h_1} \mu(\rho_{o1}^* \phi_1^*) d\xi + \int_{h_1}^{h_2} \mu(\rho_{o2}^* \phi_2^*) d\xi \tag{15.9}$$

where ϕ^* for an isotropic elastic material is given by (15.3). The explicit determination of the various coefficients in constitutive relations is beyond the scope of this project and is left for a follow-on project.

16. Constitutive coefficients for an initially flat composite laminate in bending

Using the procedure of section 15, in this section we obtain explicit forms for constitutive equations in linear theory of composite laminates with initially flat thin plies of uniform thickness. Here we confine our attention to composite laminates composed of alternating layers of only two elastic materials, each of which is assumed to be homogeneous and isotropic. It should be mentioned that any derivation of constitutive equations from three dimensional theory involves some approximations and special assumptions.

Prior to the calculation of an explicit form for the constitutive equation we need to dispose of certain preliminaries. To this end we consider a composite laminate and assume that the position vector $\mathbf{P}^*$, of the micro-body $\mathcal{B}^*$, in a reference configuration is given by

$$\mathbf{P}^* = \mathrm{R}(\eta^\alpha,\theta^3) + \xi\mathbf{D}(\eta^\alpha,\theta^3) \tag{16.1}$$

We recall that in general $\mathbf{D}$ in (1) is a three-dimensional vector having components D^1,D^2,D^3 in the direction of $\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3$. However, in the reference configuration without loss of generality we may specify $\mathbf{D}$ by

$$\mathbf{D} = \mathrm{D}\mathbf{A}_3 \quad , \quad \mathrm{D}_\alpha = 0 \quad , \quad \mathrm{D}_3 = \mathrm{D}(\eta^\alpha,\theta^3) \tag{16.2}$$

where $\mathbf{A}_3 = \mathbf{A}_3(\eta^\alpha)$ is the unit normal to the Cosserat surface, i.e., shell-like micro-structure at composite particle P. Here we make the further assumption that the position vector $\mathbf{R}$ and the director $\mathbf{D}$ in the reference configuration are given by

$$\mathbf{R}(\theta^\alpha,\theta^3) = \bar{\mathbf{R}}(\theta^\alpha) + \theta^3\mathbf{A}_3 \tag{16.3}$$

$$\mathbf{D} = \mathbf{A}_3 \quad , \quad \mathrm{D}_\alpha = 0 \quad , \quad \mathrm{D}_3 = \mathrm{D} = 1 \tag{16.4}$$

In view of $(16.4)_3$ we have

$$\xi = \zeta \tag{16.5}$$

in the reference configuration. Hence (1) may be written

$$P^* = \mathbf{R}(\eta^\alpha,\theta^3) + \zeta A_3(\eta^\alpha,\theta^3) \tag{16.6}$$

The base vectors G_i^* (i = 1,2,3) are obtained from (16.6) and are given by

$$G_\alpha^* = \mu_\alpha^\gamma A_\gamma \quad , \quad G_3^* = A_3 \tag{16.7}$$

where

$$\mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha - \zeta B^\gamma{}_\alpha \tag{16.8}$$

In the case of a flat plate $B^\gamma{}_\alpha = 0$ and (16.7) reduces to

$$G_\alpha^* = A_\alpha \quad , \quad G_3^* = A_3 \tag{16.9}$$

Recall the Gibbs function ϕ^* for an initially homogeneous and isotropic elastic material, i.e.,

$$\rho_o^* = \{- \frac{1+\nu^*}{2E^*} G_{im}^* G_{jn}^* + \frac{\nu^*}{2E^*} G_{ij}^* G_{mn}^*\}\tau^{*ij}\tau^{*mn} \tag{16.10}$$

We now expand each term on the right-hand side of (16.10) considering the second term. Thus, after making use of (16.9) and corresponding expressions for G_{ij}^*, we obtain

$$G_{ij}^* G_{mn}^* \tau^{*ij}\tau^{mn} = A_{\alpha\beta}A_{\gamma\delta}\tau^{*\alpha\beta}\tau^{*\gamma\delta} + 2A_{\alpha\beta}\tau^{*\alpha\beta}\tau^{*33} + (\tau^{*33})^2 \tag{16.11}$$

and

$$G^*_{im}G^*_{jn}\tau^{*ij}\tau^{*mn} = A_{\alpha\gamma}A_{\beta\delta}\tau^{*\alpha\beta}A^{*\gamma\delta} + 2A_{\alpha\beta}\tau^{*\alpha 3}\tau^{*\beta 3} + (\tau^{*33}) \quad (16.12)$$

Substituting (16.11) and (16.12) into (16.10), we obtain

$$\rho^*_o\phi^* = \frac{1}{2E^*}\{q_1\nu^*A_{\alpha\beta}A_{\gamma\delta} - q_2(1+\nu^*)A_{\alpha\gamma}A_{\beta\delta}\}\tau^{*\alpha\beta}\tau^{*\gamma\delta}$$

$$+ \frac{1}{E^*}A_{\alpha\beta}\{q_3\nu^*\tau^{*\alpha\beta}\tau^{*33} - q_4(1+\nu^*)\tau^{*\alpha 3}\tau^{*\beta 3}$$

$$- \frac{1}{2E^*}q_5(\tau^{*33})^2 \quad (16.13)$$

Coefficients q_1 to q_5 are assumed to be constants and may be used as tracers for each terms or calibrating factors if such need arises. Otherwise these coefficients can be put equal to unity. We call these "calibration coefficients."

Introducing (16.13) into (15.4) and recalling that for an initially flate plate we have $\mu^\gamma{}_\alpha = \delta^\gamma{}_\alpha$ we obtain

$$\rho_o\phi = q_1 A_{\alpha\beta}A_{\gamma\delta}\left(\int_o^{h_1} \frac{\nu_1^*}{2E_1^*}\tau^{*\alpha\beta}\tau^{*\gamma\delta}d\zeta + \int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{2E_2^*}\tau^{*\alpha\beta}\tau^{*\gamma\delta}d\zeta\right)$$

$$- q_2 A_{\alpha\gamma}A_{\beta\delta}\left(\int_o^{h_1} (1+\nu_1^*)\right)$$

$$- q_3 A_{\alpha\beta}\left(\int_o^{h_1} \frac{\nu_1^*}{E_1^*}\tau^{*\alpha\beta}\tau^{*33}d\zeta + \int_{h_1}^{h_1+h_2} \frac{\nu_2^*}{E_2^*}\tau^{*\alpha\beta}\tau^{*33}d\zeta\right)$$

$$- q_4 A_{\alpha\beta}\left(\int_o^{h_1} \frac{1+\nu_1^*}{E_1^*}\tau^{*\alpha 3}\tau^{*\beta 3}d\zeta + \int_{h_1}^{h_1+h_2} \frac{1+\nu_2^*}{E_2^*}\tau^{*\alpha 3}\tau^{*\beta 3}d\zeta\right)$$

$$- q_5\left(\int_o^{h_1} \frac{1}{2E_1^*}(\tau^{*33})^2 d\zeta + \int_{h_1}^{h_1+h_2} \frac{1}{2E_2^*}(\tau^{*33})^2 d\zeta\right) \quad (16.14)$$

In order to calculate an expression for ϕ in the form (16.14), we need to introduce suitable assumptions for the stress τ^{*ij} in terms of composite stress τ^{ij}, composite couple stress s^{ij} and composite intrinsic force t^i defined in (16.14).

Since the two-dimensional equations governing the behavior of the micro-structure separate into those for bending and extensional cases, it is instructive to carry out the calculation for ϕ in two parts. Thus, employing again the same symbol for a function and its value, we write

$$\phi = \phi_b + \phi_e \quad (16.15)$$

where ϕ_b and ϕ_e are associated with the bending and extensional cases, respectively. We therefore proceed to calculate the expressions for $\rho_o\phi_b$ and $\rho_o\phi_e$.

Considering the case of bending, we introduce the following expressions for stresses

$$\tau^{*\alpha\beta} = \left(\frac{6s^{\alpha\beta}}{(h_1+h_2)^2}\right)\left(\frac{\zeta-h_1}{(h_1+h_2)/2}\right) = \frac{12s^{\alpha\beta}}{(h_1+h_2)^3}(\zeta-h_1)$$

$$\tau^{*\alpha 3} = \tau^{*3\alpha} = \left(\frac{3\tau^{\alpha 3}}{2(h_1+h_2)}\right)\left(1-\left(\frac{\zeta-h_1}{(h_1+h_2)/2}\right)^2\right)$$

$$= \frac{3}{2(h_1+h_2)}\tau^{\alpha 3}\left(1-\left(\frac{2(\zeta-h_1)}{(h_1+h_2)}\right)^2\right) \qquad (16.16)$$

$$\tau^{**} = 0$$

Introducing (16.16) into (16.14) after a lengthy process we obtain

$$\rho_o\phi = q_1\frac{24}{h^3}\left(\frac{\nu_1^*}{E_1^*}n_1^3 + \frac{\nu_2^*}{E_2^*}n_2^3\right)A_{\alpha\beta}A_{\gamma\delta}s^{\alpha\beta}s^{\gamma\delta}$$

$$-q_2\frac{24}{h^3}\left(\frac{1+\nu_1^*}{E_1^*}n_1^3 + \frac{1+\nu_2^*}{E_3^*}n_2^3\right)A_{\alpha\gamma}A_{\beta\delta}s^{\alpha\beta}s^{\gamma\delta}$$

$$-q_4\frac{9}{4h}\Big[\left(n_1 - \frac{8}{3}n_1^3 + \frac{16}{5}n_1^5\right)\frac{1+\nu_1^*}{E_1^*}$$

$$+\left(n_2 - \frac{8}{3}n_2^3 + \frac{16}{5}n_2^5\right)\frac{1+\nu_2^*}{E_2^*}\Big]A_{\alpha\beta}\tau^{\alpha 3} \qquad (16.17)$$

where we have introduced the following notations

$$h = h_1 + h_2 \quad , \quad n_\alpha = \frac{h_\alpha}{h} \quad , \quad (\alpha = 1,2) \quad , \quad n = \frac{h_2}{h_1}$$

For simplicity we introduce

$$C_1 = q_1 \frac{24}{h^3} \left(\frac{\nu_1^*}{E_1^*} n_1^3 + \frac{\nu_2^*}{E_2^*} n_2^3\right)$$

$$C_2 = q_2 \frac{24}{h^3} \left(\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3\right)$$

$$C_4 = q_4 \frac{9}{4h} \left[\left(\frac{1+\nu_1^*}{E_1^*} n_1 + \frac{1+\nu_2^*}{E_2^*} n_2\right) - \frac{8}{3} \left(\frac{1+\nu_1^*}{E_1^*} + \frac{1+\nu_2^*}{E_2^*} n_8^3\right) + \frac{16}{5} \left(\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_2^*} n_3^5\right)\right] \tag{16.18}$$

Making use of (16.18) we may write (16.17) as follows

$$\rho_o = (C_1 A_{\alpha\beta} A_{\gamma\delta} - C_2 A_{\alpha\gamma} A_{\beta\delta}) s^{\alpha\beta} s^{\gamma\delta} - C_4 A_{\alpha\beta} \tau^{\alpha 3} \tau^{\beta 3} \tag{16.19}$$

17. Constitutive coefficients for an initially flat composite laminate in extension

Consider the assumptions and development in section 16, we proceed to obtain the relevant constitutive coefficients for an initially flat composite laminate in extension. Guided by the theory of Cosserat surface we introduce the following expressions for stresses

$$\tau^{*\alpha\beta} = \frac{\tau^{\alpha\beta}}{h_1+h_2}$$

$$\tau^{*\alpha 3} = \tau^{*3\alpha} = \frac{15}{(h_1+h_2)^2}\, s^{\alpha 3}\, \{\frac{2(\zeta-h_1)}{(h_1+h_2)} - (\frac{2(\zeta-h_1)}{(h_1+h_2)})^3\} \qquad (17.1)$$

$$\tau^{*33} = \frac{k^3}{h_1+h_3}$$

Introducing (17.1) into (16.14) we obtain:

$$\rho_o\phi = q_1\,\frac{1}{2h}\,(\frac{\nu_1^*}{E_1^*}\,n_1 + \frac{\nu_2^*}{E_2^*}\,n_2)A_{\alpha\beta}A_{\gamma\delta}\tau^{\alpha\beta}\tau^{\gamma\delta}$$

$$-\,q_2\,\frac{1}{2h}!(\frac{1+\nu_1^*}{E_2^*}\,n_1 + \frac{1+\nu_2^*}{E_2^*}\,n_2)A_{\alpha\gamma}A_{\beta\delta}\tau^{\alpha\beta}\tau^{\gamma\delta}$$

$$-\,q_3\,\frac{1}{h}\,(\frac{\nu_1^*}{E_1^*}\,n_1 + \frac{\nu_2^*}{E_2^*}\,n_2)A_{\alpha\beta}\tau^{\alpha\beta}k^3$$

$$-\,q_4\,\frac{900}{h^3}\,[\frac{1}{3}\,(\frac{1+\nu_1^*}{E_1^*}\,n_1^3 + \frac{1+\nu_2^*}{E_2^*}\,n_2^3)$$

$$- \frac{8}{5} \left(\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_2^*} n_2^5 \right)$$

$$+ \frac{16}{7} \left(\frac{1+\nu_1^*}{E_1^*} n_1^7 + \frac{1+\nu_2^*}{E_2^*} n_2^7 \right) \Big] A_{\alpha\beta} s^{\alpha 3} s^{\beta 3}$$

$$- q_5 \frac{1}{2h} \left(\frac{1}{E_1^*} n_1 + \frac{1}{E_2^*} n_2 \right) (k^3)^2 \qquad (17.2)$$

For simplicity we introduce

$$D_1 = q_1 \frac{1}{2h} \left(\frac{\nu_1^*}{E_1^*} n_1 + \frac{n_2^*}{E_2^*} n_2 \right)$$

$$D_2 = q_2 \frac{1}{2h} \left(\frac{1+\nu_1^*}{E_1^*} n_1 + \frac{1+\nu_2^*}{E_2^*} n_2 \right)$$

$$D_3 = q_3 \frac{1}{h} \left(\frac{\nu_1^*}{E_1^*} n_1 + \frac{\nu_2^*}{E_2^*} n_2 \right) \qquad (17.3)$$

$$D_4 = q_4 \frac{900}{h^3} \Big[\frac{1}{3} \left(\frac{1+\nu_1^*}{E_1^*} n_1^3 + \frac{1+\nu_2^*}{E_2^*} n_2^3 \right)$$

$$- \frac{8}{5} \left(\frac{1+\nu_1^*}{E_1^*} n_1^5 + \frac{1+\nu_2^*}{E_2^*} n_2^5 \right)$$

$$+ \frac{16}{7} \left(\frac{1+\nu_1^*}{E_1^*} n_1^7 + \frac{1+\nu_2^*}{E_2^*} n_2^7 \right) \Big]$$

$$D_5 = q_5 \frac{1}{2h} \left(\frac{1}{E^{1*}} n_1 + \frac{1}{E_2^*} n_2\right)$$

$$\rho_o \phi = (D_1 A_{\alpha\beta} A_{\gamma\delta} - D_2 A_{\alpha\gamma} A_{\beta\delta}) \tau^{\alpha\beta} \tau^{\gamma\delta} + D_3 A_{\alpha\beta} \tau^{\alpha\beta} k^3$$

$$- D_4 A_{\alpha\beta} s^{\alpha 3} s^{\beta 3} - D_5 (k^3)^2 \qquad (17.4)$$